JAMAICA COLLEGE

C.S.E.C. MATHEMATICS MARATHON (PART TWO)

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RELATIONS, FUNCTIONS & GRAPHS

A function is a relation in which each element in the domain is mapped to only ONE element in the co-domain.

So, one-to-one and many-to-one relations are functions while one-to-many relations are NOT functions.

A function is a series of operations in sequence.
A flowchart of a function is as follows:

INPUT ⟶ SERIES OF OPERATIONS ⟶ OUTPUT

An example of a function is $f : x \to 4x - 5$
Here, $f(x) = 4x - 5$
Now, $f(x) = y$
So, $y = 4x - 5$

Now, if $f(x) = 4x - 5$, then

$$f(0) = 4(0) - 5$$
$$= 0 - 5$$
$$\therefore \quad f(0) = -5$$

Now, if $g(x) = 7 - 2x^2$, then

$$g(-3) = 7 - 2(-3)^2$$
$$= 7 - 2(9)$$
$$= 7 - 18$$
$$\therefore \quad g(-3) = -11$$

Now, if $h(x) = \dfrac{4-x}{3x+1}$, then

$$h\left(\frac{1}{2}\right) = \frac{4 - \frac{1}{2}}{3\left(\frac{1}{2}\right) + 1} \quad = \frac{4 - \frac{1}{2}}{\frac{3}{2} + 1}$$

$$=\frac{\left(\frac{8}{2}-\frac{1}{2}\right)}{\left(\frac{3}{2}+\frac{2}{2}\right)}$$

$$=\frac{\left(\frac{7}{2}\right)}{\left(\frac{5}{2}\right)} \quad = \frac{7}{2}\times\frac{2}{5}$$

$$\therefore \quad h\left(\frac{1}{2}\right)=\frac{7}{5}$$

Composite functions

If $f(x)=2x-1$ and $g(x)=7x+5$, then $fg(x)$ is found by replacing x in $f(x)$ with $7x+5$.

Similarly, $gf(x)$ is found by replacing x in $g(x)$ with $2x-1$.

Now, $fg(x)=f(7x+5)$

$$=2(7x+5)-1$$
$$=14x+10-1$$
$$=14x+9$$

Also, $gf(x)=g(2x-1)$

$$=7(2x-1)+5$$
$$=14x-7+5$$
$$gf(x)=14x-2$$

Consider the functions $f(x)=1-x^2$ and $g(x)=4x+9$.

Now, $fg(x)=f(4x+9)$

$$=1-(4x+9)^2$$
$$=1-[(4x+9)(4x+9)]$$
$$=1-(16x^2+36x+36x+81)$$
$$=1-(16x^2+72x+81)$$
$$=1-16x^2-72x-81$$
$$=-16x^2-72x-81+1$$
$$fg(x)=-16x^2-72x-80$$

Also, $gf(x)=g(1-x^2)$

$$=4(1-x^2)+9$$

$= 4 - 4x^2 + 9$

$= -4x^2 + 4 + 9$

$gf(x) = -4x^2 + 13$

Consider the functions $f(x) = 2x - 1$, $g(x) = x^2$ and $h(x) = 3 + 5x$

Now, $hfg(x) = hf(x^2)$

$= h\left[2(x^2) - 1\right]$

$= h(2x^2 - 1)$

So, $hfg(x) = h(2x^2 - 1)$

Now, $hfg(x) = 3 + 5(2x^2 - 1)$

$= 3 + 10x^2 - 5$

$= 10x^2 - 5 + 3$

$hfg(x) = 10x^2 - 2$

Finding the inverse of a function

If $f(x)$ is a function, then its INVERSE is denoted by $f^{-1}(x)$ ('f inverse of x').

It means that the inverse of $g(x)$ is $g^{-1}(x)$.

Also, the inverse of $h(x)$ is $h^{-1}(x)$.

Steps for finding the inverse of a function

(i) Replace $f(x)$ with y $\left(\text{OR } g(x) \text{ with } y \text{ OR } h(x) \text{ with } y \text{ etc}\right)$

(ii) INTERCHANGE x AND y and make y the subject of the resulting equation

(iii) Label the resulting equation as $f^{-1}(x)$ $\left(\text{OR } g^{-1}(x) \text{ OR } h^{-1}(x) \text{ etc}\right)$

Consider the function $f : x \to 2x + 3$

Here, $f(x) = 2x + 3$

Now, $y = 2x + 3$

Interchanging x and y gives $x = 2y + 3$

Now, $2y = x - 3$

So, $y = \dfrac{x - 3}{2}$

$\therefore \quad f^{-1}(x) = \dfrac{x - 3}{2}$

Example

Find the inverse of the function $g(x) = 4 + 3x$

Solution

Here, $g(x) = 4 + 3x$

Now, $y = 4 + 3x$

Interchanging x and y gives $x = 4 + 3y$

Now, $x - 4 = 3y$

So, $3y = x - 4$

So, $y = \frac{x-4}{3}$

$\therefore \quad g^{-1}(x) = \frac{x-4}{3}$

Example

(a) Find $h^{-1}(x)$ if $h(x) = 4x^2 - 11$

(b) Hence, find $h^{-1}(5)$

Solution

(a) Let $y = 4x^2 - 11$

Interchanging x and y gives $x = 4y^2 - 11$

Now, $x + 11 = 4y^2$

So, $4y^2 = x + 11$

Now, $y^2 = \frac{x+11}{4}$

So, $y = \sqrt{\frac{x+11}{4}}$

$\therefore \quad h^{-1}(x) = \sqrt{\frac{x+11}{4}}$

(b) Now, $h^{-1}(5) = \sqrt{\frac{5+11}{4}}$

$= \sqrt{\frac{16}{4}}$

$= \sqrt{4}$

$= 2$

Example

Find $f^{-1}(x)$ if $f(x)=\dfrac{5x+3}{2x-1},\ x\neq\dfrac{1}{2}$

Solution

Let $y=\dfrac{5x+3}{2x-1}$

Interchanging x and y gives $x=\dfrac{5y+3}{2y-1}$

Now, $\dfrac{x}{1}=\dfrac{5y+3}{2y-1}$

Now, by cross-multiplying, we get $x(2y-1)=5y+3$

So, $2xy-x=5y+3$

So, $2xy-5y=x+3$

Now, $y(2x-5)=x+3$

So, $y=\dfrac{x+3}{2x-5}$

$\therefore\ f^{-1}(x)=\dfrac{x+3}{2x-5},\ x\neq\dfrac{5}{2}$

Consider the functions $f(x)=5x-7$ and $g(x)=\dfrac{x+7}{5}$

Now, $f[g(x)]=f\left(\dfrac{x+7}{5}\right)$

$=5\left(\dfrac{x+7}{5}\right)-7$

$=x+7-7$

$f[g(x)]=x$

Also, $g[f(x)]=g(5x-7)$

$=\dfrac{5x-7+7}{5}$

$=\dfrac{5x}{5}$

$g[f(x)]=x$

Since $f[g(x)]=x$ AND $g[f(x)]=x$, then the functions $f(x)$ and $g(x)$ are inverses of each other!!!

Example (May 2019)

The functions f and g are defined by

$$f(x)=\frac{9}{2x+1} \text{ and } g(x)=x-3$$

(a) State a value of x that CANNOT be in the domain of f (1 mark)

(b) Find, in its simplest form, expressions for

i) $fg(x)$ (2 marks)

ii) $f^{-1}(x)$ (2 marks)

Solution

(a) If $2x+1=0$, then $2x=-1$

So, $x=-\frac{1}{2}$

$\therefore$ the value of x that CANNOT be in the domain of f is $-\frac{1}{2}$

(b) $fg(x)=\dfrac{9}{2(x-3)+1}$

$$=\frac{9}{2x-6+1}$$

$$=\frac{9}{2x-5}$$

(c) Let $y=\dfrac{9}{2x+1}$

Interchanging x and y gives $x=\dfrac{9}{2y+1}$

Now, $\dfrac{x}{1}=\dfrac{9}{2y+1}$

Now, $x(2y+1)=9$

So, $2y+1=\dfrac{9}{x}$

Now, $2y = \frac{9}{x} - 1$

Now, $y = \frac{1}{2}\left(\frac{9}{x} - 1\right)$

So, $y = \frac{9}{2x} - \frac{1}{2} \quad = \frac{9-x}{2x}$

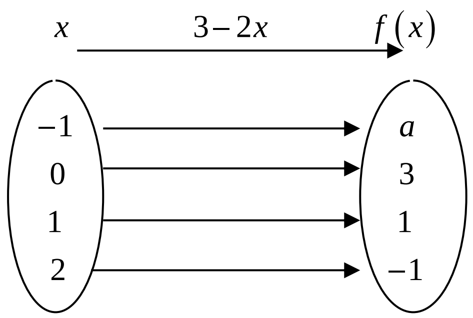
$\therefore \quad f^{-1}(x) = \frac{9-x}{2x}$

Example (January 2021)

The functions f is defined as

$$f : x \rightarrow 3 - 2x$$

(a) The diagram below shows the mapping diagram of the function, f.



$a =$ .... (1 mark)

(b) Determine, in their simplest forms, expressions for

i) the inverse of the function f, $f^{-1}(x)$ (1 mark)

ii) the composite function $f^{2}(x)$ (2 marks)

(c) State the value of $ff^{-1}(-2)$ (1 mark)

Solution

(a) Here, $f(x) = 3 - 2x$

Now, $f(-1) = 3 - 2(-1) \quad = 3 + 2$

So, $f(-1) = 5$

$\therefore \quad a = 5$

(b) i) Let $y = 3 - 2x$

Interchanging x and y gives $x = 3 - 2y$

Now, $2y = 3 - x$

So, $y = \dfrac{3-x}{2}$

$\therefore \quad f^{-1}(x) = \dfrac{3-x}{2}$

ii) $f^2(x) = ff(x)$

$= 3 - 2(3 - 2x)$

$= 3 - 6 + 4x$

$= 4x - 3$

(c) Since $f(x) = 3 - 2x$ and $f^{-1}(x) = \dfrac{3-x}{2}$, then

$$ff^{-1}(x) = 3 - 2\left(\frac{3-x}{2}\right)$$

$$= 3 - (3 - x)$$

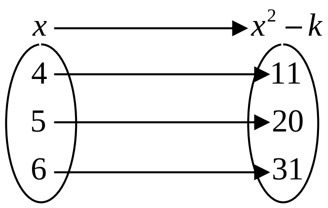
$$= 3 - 3 + x$$

$$= x$$

Now, $ff^{-1}(-2) = -2$

TRY THIS (January 2011)

1. The arrow diagram shown below represents the relation $f : x \to x^2 - k$, where $x \in \{3, 4, 5, 6, 7, 8, 9, 10\}$



Calculate the value of

(a) k (2 marks)

(b) $f(3)$ (2 marks)

(c) x when $f(x)=95$ (2 marks)

Example (May 2013)

Given that $f(x)=\dfrac{2x+1}{3}$ and $g(x)=4x+5$, determine the values of

(a) $fg(2)$ (3 marks)

(b) $f^{-1}(3)$ (3 marks)

Solution

(a) $fg(x)=\dfrac{2(4x+5)+1}{3}$

$=\dfrac{8x+10+1}{3}$

$fg(x)=\dfrac{8x+11}{3}$

Now, $fg(2)=\dfrac{8(2)+11}{3}$

$=\dfrac{16+11}{3} \quad =\dfrac{27}{3}$

$fg(2)=9$

(b) Let $y=\dfrac{2x+1}{3}$

Interchanging x and y gives $x=\dfrac{2y+1}{3}$

Now, $\dfrac{x}{1}=\dfrac{2y+1}{3}$

Now, $2y+1=3x$

So, $2y=3x-1$

So, $y=\dfrac{3x-1}{2}$

$\therefore \quad f^{-1}(x)=\dfrac{3x-1}{2}$

Now, $f^{-1}(3) = \frac{3(3)-1}{2}$

$= \frac{9-1}{2} = \frac{8}{2}$

$f^{-1}(3) = 4$

TRY THIS (May 2022)

2. The functions f and g are defined as follows:

$f(x) = 5x+7$ and $g(x) = 3x-1$.

For the functions given above, determine

(a) $g\left(\frac{1}{3}\right)$ (1 mark)

(b) $f^{-1}(-3)$ (2 marks)

<u>Example (May 2023)</u>

Consider the following functions.

$$f(x) = \frac{3}{x+2}, \quad g(x) = 4x-5 \quad \text{and} \quad h(x) = x^2+1$$

(a) (i) For what value of x is $f(x)$ undefined? (1 mark)

(ii) Find the value of

a) $g\left(\frac{1}{4}\right)$ (1 mark)

b) $h(-3)$ (1 mark)

c) $ff(0)$ (2 marks)

(b) Write an expression, in its SIMPLEST form, for $gh(x)$ (2 marks)

(c) Find $g^{-1}(-2)$ (2 marks)

Solution

(a) (i) When $x+2=0$, $x=-2$

$\therefore$ $f(x)$ undefined when $x=-2$

(ii) a) $g\left(\frac{1}{4}\right)=4\left(\frac{1}{4}\right)-5$

$=1-5$

$=-4$

b) $h(-3)=(-3)^2+1$

$=9+1$

$=10$

c) Since $f(x)=\frac{3}{x+2}$, then $f(0)=\frac{3}{0+2}$

So, $f(0)=\frac{3}{2}$

Now, $ff(0)=f\left(\frac{3}{2}\right)$

$=\frac{3}{\frac{3}{2}+2}$ $=\frac{3}{\left(\frac{3}{2}+\frac{4}{2}\right)}$

$=\frac{3}{\left(\frac{7}{2}\right)}$

$=\frac{3}{1}\div\frac{7}{2}$

$=\frac{3}{1}\times\frac{2}{7}$

$=\frac{6}{7}$

(b) $gh(x)=g\left(x^2+1\right)$

$=4\left(x^2+1\right)-5$

$=4x^2+4-5$

$=4x^2-1$

(c) Firstly, we need to find $g^{-1}(x)$

Since $g(x) = 4x - 5$, then $g^{-1}(x) = \frac{x+5}{4}$

Now, $g^{-1}(-2) = \frac{-2+5}{4}$

$\therefore \quad g^{-1}(-2) = \frac{3}{4}$

Table of values

A table of values may be constructed to show corresponding domain and co-domain values for a function.

Consider the function $f: x \to 4x - 5$ for the domain $-3 \leq x \leq 5$

The table of values is as follows:

x	-3	-2	-1	0	1	2	3	4	5
$4x$	-12	-8	-4	0	4	8	12	16	20
-5	-5	-5	-5	-5	-5	-5	-5	-5	-5
$f(x)$	-17	-13	-9	-5	-1	3	7	11	15

The graph of the function is shown below.

Example

Draw the graph of the function $f: x \to 2x + 1$ for the domain $-3 \leq x \leq 2$

Solution

x	-3	-2	-1	0	1	2
$2x$	-6	-4	-2	0	2	4
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
$f(x)$	-5	-3	-1	1	3	5

The graph of the function is shown below.

Note: The functions $f(x) = 4x - 5$ and $f(x) = 2x + 1$ are LINEAR functions.
So, a STRAIGHT LINE is used to join the points on the graph sheet.

Graph of $f(x) = 4x - 5$, $-3 \leq x \leq 5$

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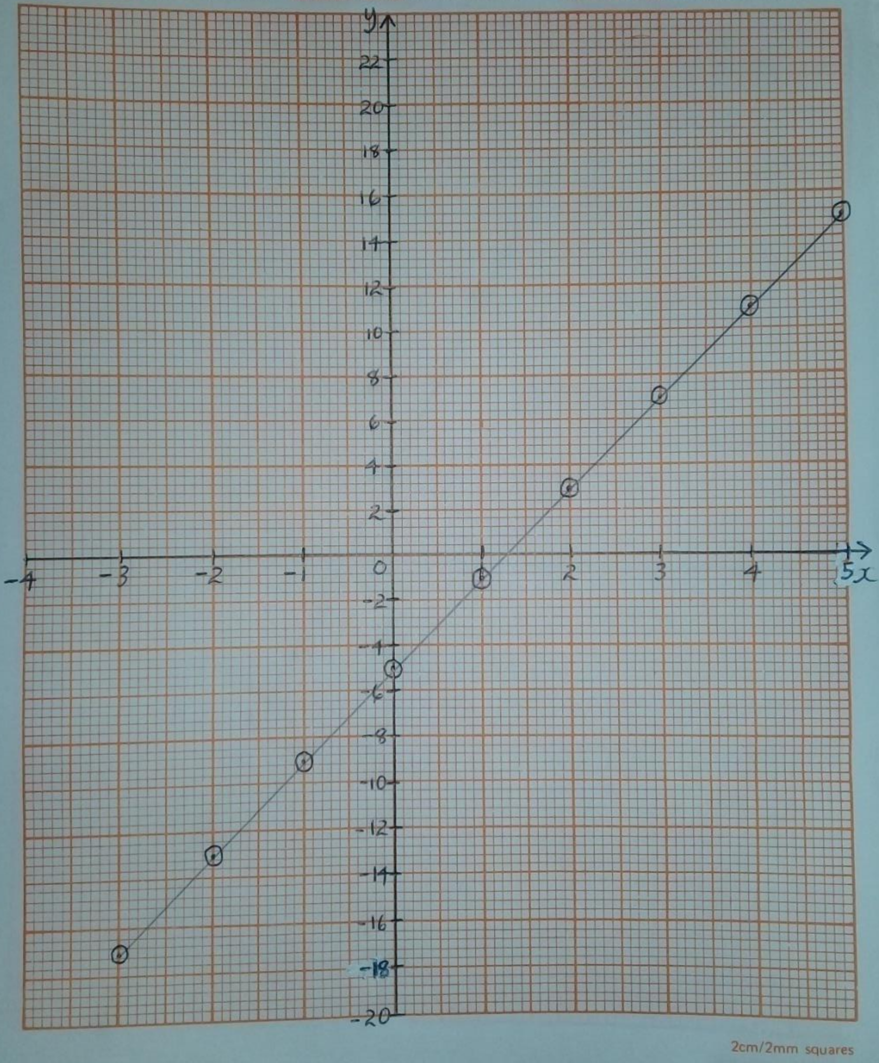

2cm/2mm squares

Graph of $f(x) = 2x+1$, $-3 \leq x \leq 2$

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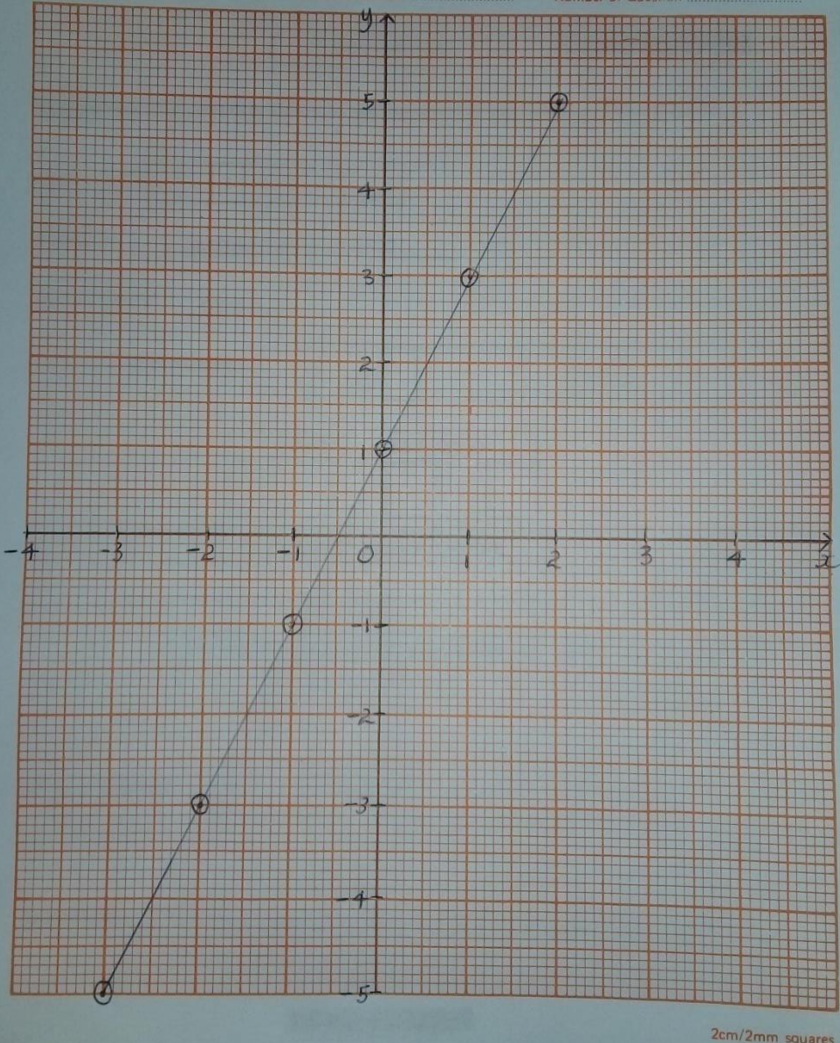


2cm/2mm squares

Example

Draw the graph of the function $f : x \to x^2 + 2x - 8$ for the domain $-5 \le x \le 3$

Solution

x	-5	-4	-3	-2	-1	0	1	2	3
x^2	25	16	9	4	1	0	1	4	9
$+2x$	-10	-8	-6	-4	-2	$+0$	$+2$	$+4$	$+6$
-8	-8	-8	-8	-8	-8	-8	-8	-8	-8
$f(x)$	7	0	-5	-8	-9	-8	-5	0	7

The graph of the function is shown below.

Example

(a) Draw the graph of the function $f : x \to -x^2 + 2x + 3$ for the domain $-2 \le x \le 4$

(b) Hence, find the roots of the equation $-x^2 + 2x + 3 = 0$

Solution

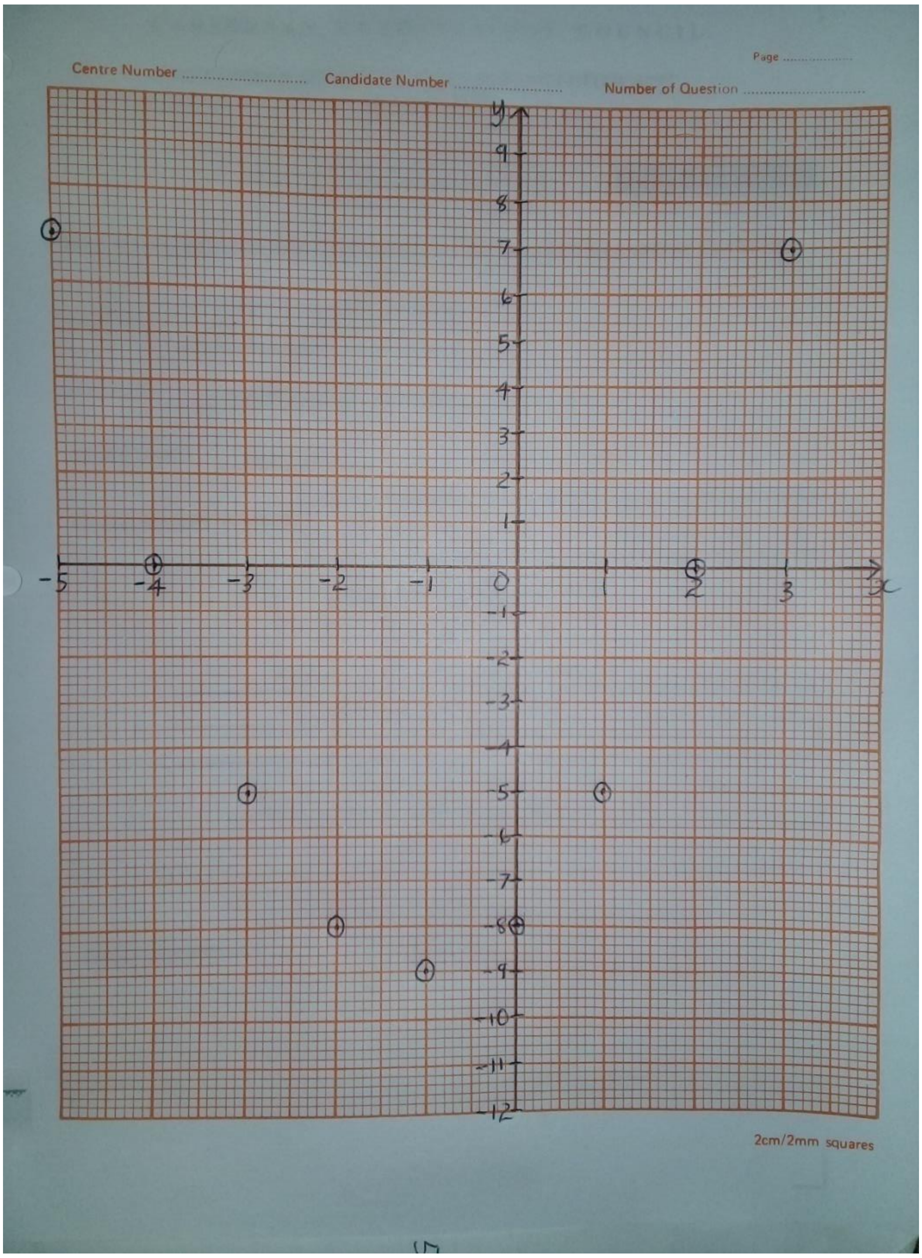
x	-2	-1	0	1	2	3	4
$-x^2$	-4	-1	0	-1	-4	-9	-16
$+2x$	-4	-2	$+0$	$+2$	$+4$	$+6$	$+8$
$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
$f(x)$	-5	0	3	4	3	0	-5

The graph of the function is shown below.

Note: The functions $f(x) = x^2 + 2x - 8$ and $f(x) = -x^2 + 2x + 3$ are QUADRATIC functions.

We CANNOT use straight lines to join the consecutive pairs of points on the graph sheet.

We MUST join the points to form a CURVE.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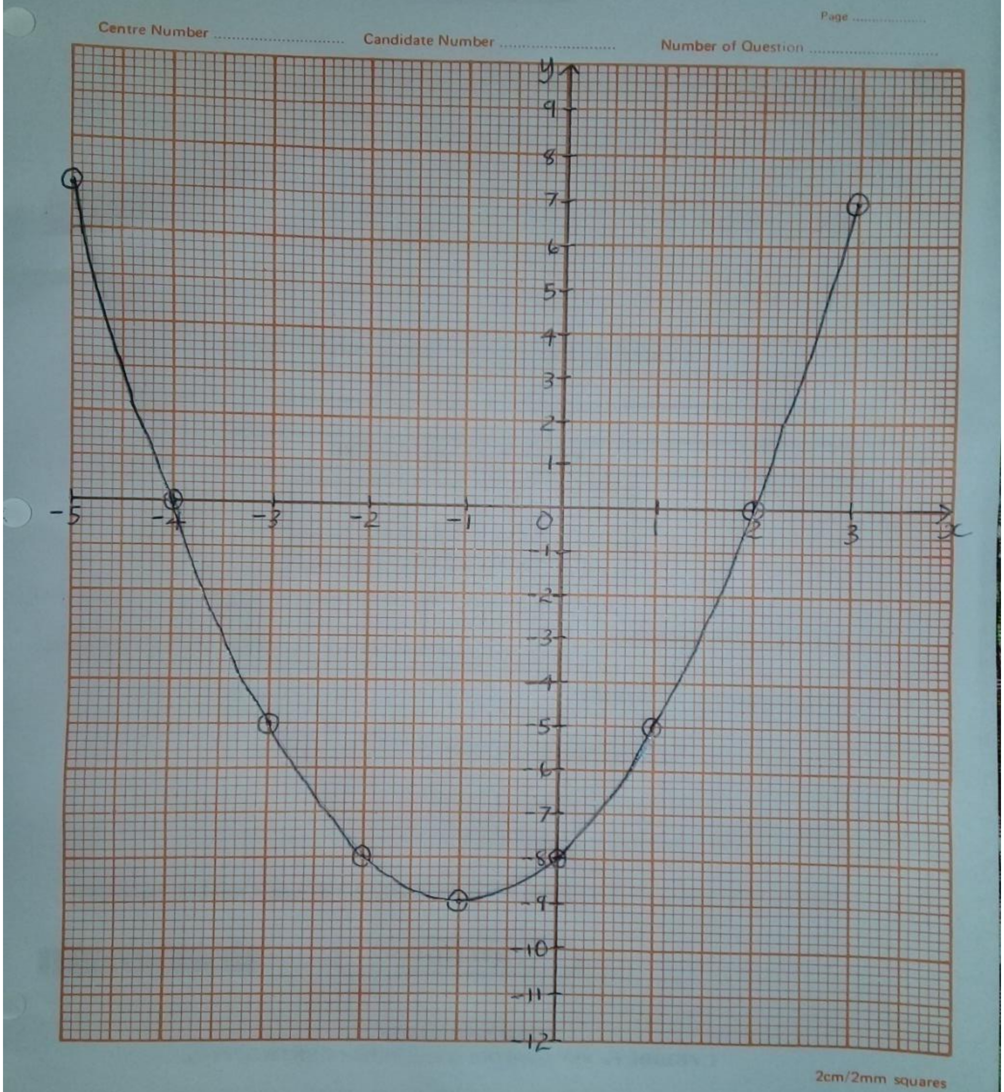
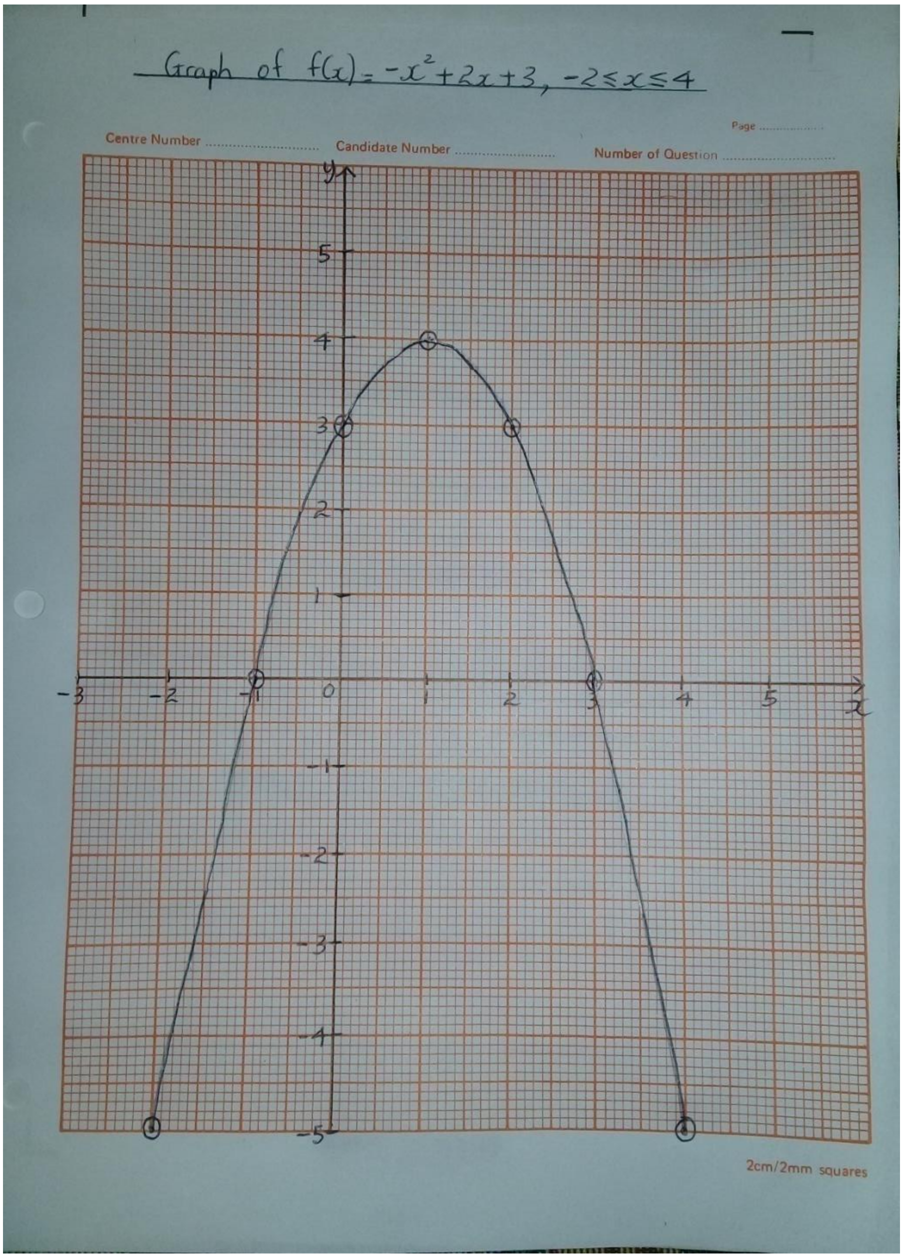
Graph of $f(x) = x^2 + 2x - 8$, $-5 \leq x \leq 3$





(b) The roots of the equation $-x^2+2x+3=0$ are -1 and 3

Example (January 2012)

The table below shows corresponding values of x and y for the function $y = x^2 - 2x - 3$, for integer values of x from –2 to 4.

x	–2	–1	0	1	2	3	4
y	5		–3	–4		0	5

(a) Copy and complete the table. **(2 marks)**

(b) **Using a scale of 2 cm to represent 1 unit on the x-axis, and 1 cm to represent 1 unit on the y-axis**, plot the points whose x and y values are recorded in your table, and draw a smooth curve through your points. **(4 marks)**

(c) Using your graph, estimate the value of y when $x = 3.5$. Show on your graph how the value was obtained. **(2 marks)**

(d) Without further calculations,

(i) write the equation of the axis of symmetry of the graph **(1 mark)**

(ii) estimate the minimum value of the function y **(1 mark)**

(iii) state the values of the solutions of the equation: $x^2 - 2x - 3 = 0$. **(1 mark)**

Total 11 marks

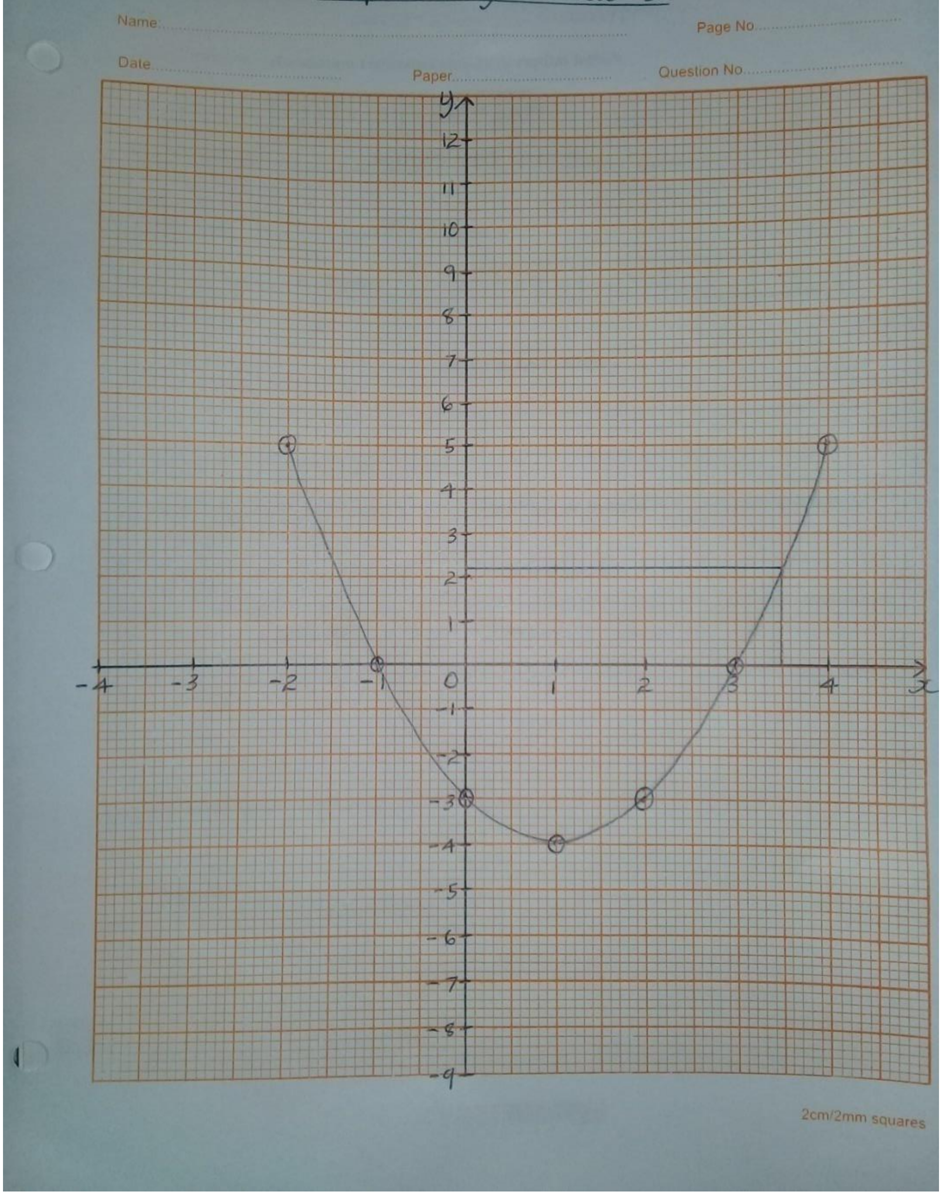
Solution

(a)

x	–2	–1	0	1	2	3	4
y	5	0	–3	–4	–3	0	5

(b) See graph below

Graph of $y = x^2 - 2x - 3$



(c) From the graph, $y = 2.2$ when $x = 3.5$

(d) i) Since the minimum point on the curve is $(1, -4)$, the axis of symmetry is the straight line $x = 1$

(d) ii) Since the minimum point on the curve is $(1, -4)$, the minimum value of y is -4

(d) iii) The solutions for the equation $x^2 - 2x - 3 = 0$ are $x = -1$ and $x = 3$

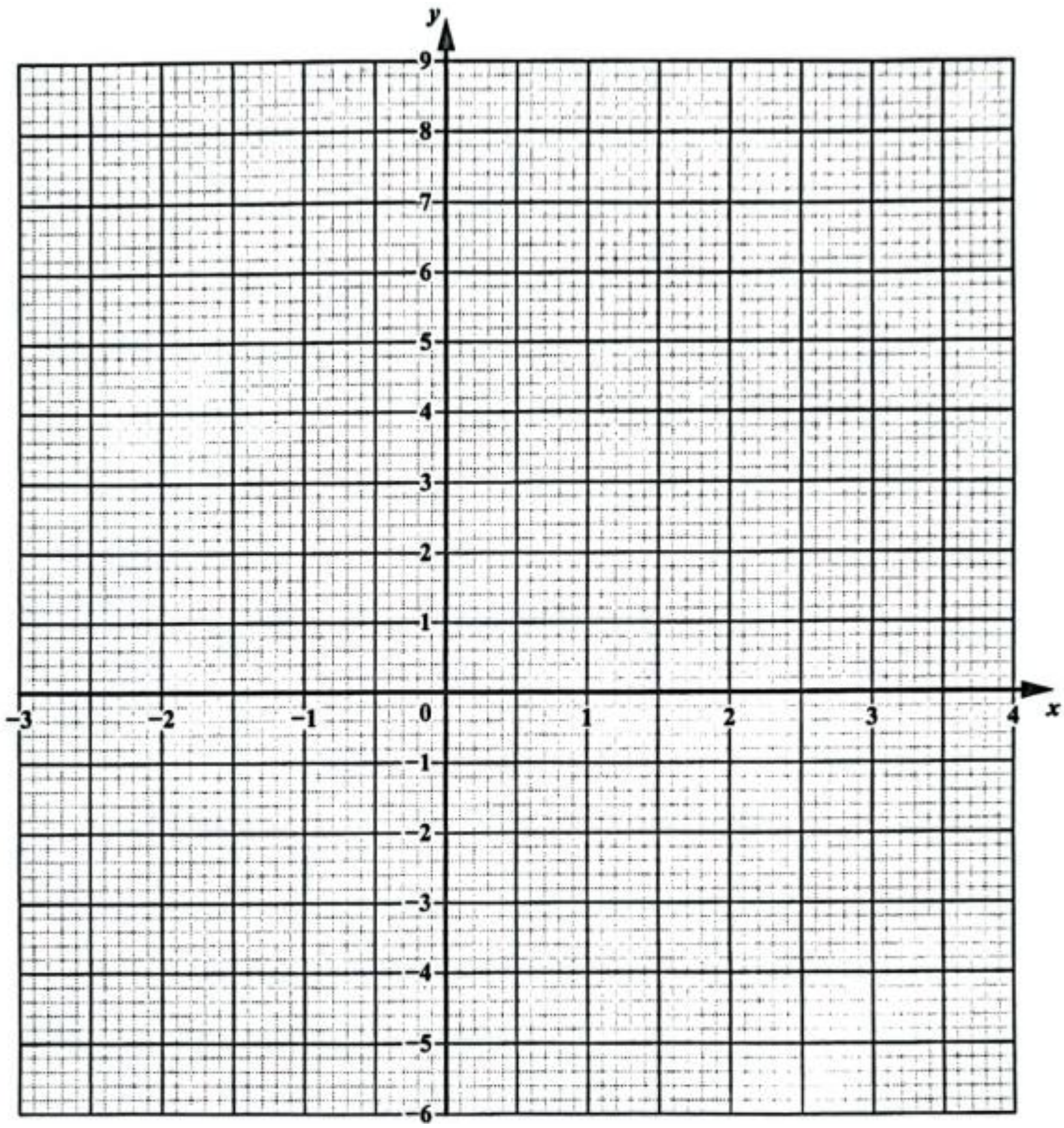
TRY THIS (May 2023)

3. (a) Complete the table for the function $y = -x^2 + x + 7$.

x	−3	−2	−1	0	1	2	3	4
y		1		7		5		−5

(2 marks)

(b) On the grid below, draw the graph of $y = -x^2 + x + 7$ for $-3 \le x \le 4$.



(3 marks)

(c) Write down the coordinates of the maximum/minimum point of the graph

(.................... , ...) (1 mark)

(d) Write down the equation of the axis of symmetry of the graph (1 mark)

(e) Use your graph to find the solutions of the equation $-x^2 + x + 7 = 0$

$x =$................................ or $x =$................................ (2 marks)

(f) (i) On the grid above, draw a line through the points $(-3, -1)$ and $(0, 8)$

(1 mark)

(ii) Determine the equation of this line in the form $y = mx + c$ (2 marks)

Completing the square

Steps:
(i) Ensure that the coefficient of the squared term is +1
(ii) Half the coefficient of the 'middle' term, square then add/subtract to the original expression
(iii) Identify the PERFECT SQUARE then group resulting constants

Consider the expression $x^2 - x + 4$.

Here, the coefficient of x^2 is $+1$

Now, $\left(\frac{1}{2} \times -1\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

Adding/subtracting $\frac{1}{4}$ to the original expression gives

$$x^2 - x + 4 = \underbrace{x^2 - x + \frac{1}{4}}{} - \frac{1}{4} + 4$$

$$= \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 4$$

$$= \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{16}{4}$$

$$\therefore \; x^2 - x + 4 = \left(x - \frac{1}{2}\right)^2 + \frac{15}{4}$$

{in the form $a(x+h)^2 + k$}

$\left(x - \frac{1}{2}\right)^2 = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$
$= x^2 - \frac{1}{2}x - \frac{1}{2}x + \frac{1}{4}$
$= x^2 - x + \frac{1}{4}$

Consider the quadratic expression $2x^2 + 5x + 3$

Now, $2x^2 + 5x + 3 = 2\left[x^2 + \frac{5}{2}x + \frac{3}{2}\right]$

Now, $\left(\frac{1}{2} \times \frac{5}{2}\right)^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$

Adding/subtracting $\frac{25}{16}$ to the original expression gives

$$2x^2 + 5x + 3 = 2\left[\underbrace{x^2 + \frac{5}{2}x + \frac{25}{16}}{} - \frac{25}{16} + \frac{3}{2}\right]$$

$$= 2\left[\left(x+\frac{5}{4}\right)^2 - \frac{25}{16} + \frac{3}{2}\right]$$

$$= 2\left[\left(x+\frac{5}{4}\right)^2 - \frac{25}{16} + \frac{24}{16}\right]$$

$$= 2\left[\left(x+\frac{5}{4}\right)^2 - \frac{1}{16}\right]$$

$$= 2\left(x+\frac{5}{4}\right)^2 - \frac{2}{16}$$

$$\left(x+\frac{5}{4}\right)^2 = \left(x+\frac{5}{4}\right)\left(x+\frac{5}{4}\right)$$

$$= x^2 + \frac{5}{4}x + \frac{5}{4}x + \frac{25}{16}$$

$$= x^2 + \frac{5}{2}x + \frac{25}{16}$$

$\therefore\ 2x^2+5x+3 = 2\left(x+\frac{5}{4}\right)^2 - \frac{1}{8}$ $\quad$ $\{$in the form $a(x+h)^2+k\}$

Consider the quadratic expression $-x^2-x+1$

Now, $-x^2-x+1 = -\left[x^2+x-1\right]$

Now, $\left(\frac{1}{2}\times 1\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

Adding/subtracting $\frac{1}{4}$ to the original expression gives

$$-x^2-x+1 = -\left[\underbrace{x^2+x+\frac{1}{4}}-\frac{1}{4}-1\right]$$

$$= -\left[\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} - 1\right]$$

$$= -\left[\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} - \frac{4}{4}\right]$$

$$= -\left[\left(x+\frac{1}{2}\right)^2 - \frac{5}{4}\right]$$

$$\left(x+\frac{1}{2}\right)^2 = \left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$$

$$= x^2 + \frac{1}{2}x + \frac{1}{2}x + \frac{1}{4}$$

$$= x^2 + x + \frac{1}{4}$$

$\therefore\ -x^2-x+1 = -\left(x+\frac{1}{2}\right)^2 + \frac{5}{4}$ $\quad$ $\{$in the form $a(x+h)^2+k\}$

$$\therefore\ -2x^2+x-9=-2\left(x-\frac{1}{4}\right)^2-\frac{71}{8} \qquad \left\{\text{in the form } a(x+h)^2+k\right\}$$

Most times quadratic expressions of the general form ax^2+bx+c are expressed in the form $a(x+h)^2+k$, where a, h and k are real numbers.
The minimum or maximum point on the graph of $y=a(x+h)^2+k$ may be known along with the minimum or maximum value of the function ax^2+bx+c.

Also, the AXIS OF SYMMETRY of the function ax^2+bx+c may be determined.
The axis of symmetry of the curve $y=ax^2+bx+c$ is the line $x=-\dfrac{b}{2a}$.

When the curve $y=ax^2+bx+c$ is written in the form $y=a(x+h)^2+k$, where a, h and k are real numbers (constants), we can determine the minimum or maximum turning point and the minimum or maximum value of the function ax^2+bx+c.

If $a>0$, the curve $y=ax^2+bx+c$ has the shape '$\cup$' and a MINIMUM turning point at the point $(-h,\ k)$.
Also, the minumum value of ax^2+bx+c is k if $a>0$.

If $a<0$, the curve $y=ax^2+bx+c$ has the shape '$\cap$' and a MAXIMUM turning point at the point $(-h,\ k)$.
Also, the maximum value of ax^2+bx+c is k if $a<0$.

Consider the curve $y=2x^2-3x+1$

Now, $y=2\left[x^2-\dfrac{3}{2}x+\dfrac{1}{2}\right]$

So, $y=2\left[\underbrace{x^2-\dfrac{3}{2}x+\dfrac{9}{16}}-\dfrac{9}{16}+\dfrac{1}{2}\right]$

$$\left(\frac{1}{2}\times-\frac{3}{2}\right)^2=\left(-\frac{3}{4}\right)^2=\frac{9}{16}$$

Now, $y=2\left[\left(x-\dfrac{3}{4}\right)^2-\dfrac{9}{16}+\dfrac{8}{16}\right]$

So, $y=2\left[\left(x-\dfrac{3}{4}\right)^2-\dfrac{1}{16}\right]$

So, $y = 2\left(x - \frac{3}{4}\right)^2 - \frac{2}{16}$

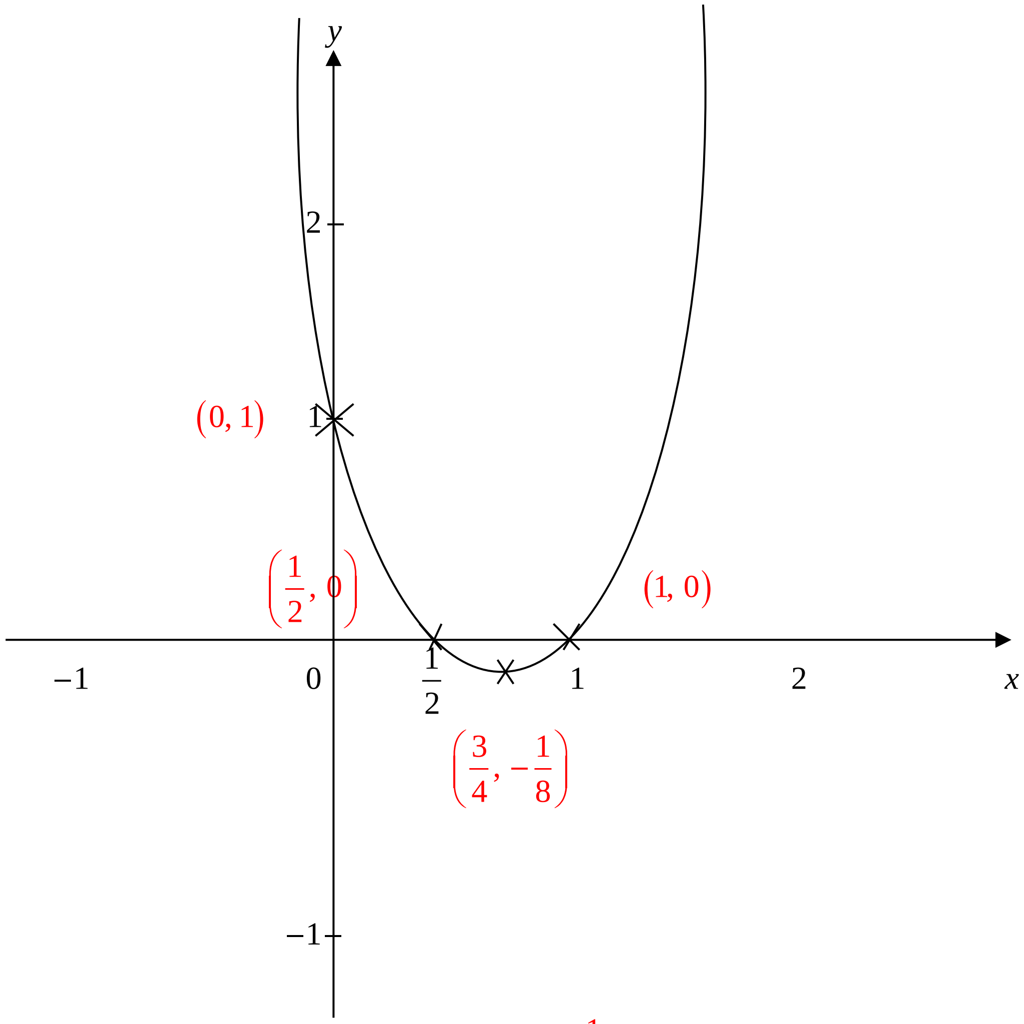
$\therefore \quad y = 2\left(x - \frac{3}{4}\right)^2 - \frac{1}{8}$ $\quad \left\{\text{in the form } y = a(x+h)^2 + k\right\}$

Since $a > 0$, then the curve $y = 2x^2 - 3x + 1$ has the shape '$\cup$' and a MINIMUM turning point at $(-h, k) = \left(\frac{3}{4}, -\frac{1}{8}\right)$.

Substituting $x = 0$ into the equation $y = 2x^2 - 3x + 1$ gives $y = 1$.
So, the curve $y = 2x^2 - 3x + 1$ cuts the $y-$axis at the point $(0, 1)$

Substituting $y = 0$ into the equation $y = 2x^2 - 3x + 1$ gives $2x^2 - 3x + 1 = 0$
Now, $2x^2 - 2x - x + 1 = 0$
So, $2x(x-1) - 1(x-1) = 0$
Now, $(x-1)(2x-1) = 0$
Now, $x - 1 = 0$ OR $2x - 1 = 0$
So, $x = 1$ OR $2x = 1$
$\therefore \; x = 1$ OR $x = \frac{1}{2}$
So, the curve $y = 2x^2 - 3x + 1$ cuts the $x-$axis at the points $\left(\frac{1}{2}, 0\right)$ and $(1, 0)$.

Sketch of $y = 2x^2 - 3x + 1$



Note: The mimimum VALUE of $2x^2 - 3x + 1$ is $-\frac{1}{8}$

The axis of symmetry of $2x^2 - 3x + 1$ is $x = -\frac{b}{2a} = -\frac{(-3)}{2(2)}$

So, the axis of symmetry of the curve $y = 2x^2 - 3x + 1$ is the line $x = \frac{3}{4}$

Past paper (January 2011)

(a) Express the quadratic function $1-6x-x^2$ in the form $k-a(x+h)^2$, where a, h and k are constants. (3 marks)

(b) Hence state

i) the maximum value of $1-6x-x^2$

ii) the equation of the axis of symmetry of the quadratic function (2 marks)

(c) Determine the roots of $1-6x-x^2=0$, giving your answers to 2 decimal places. (3 marks)

Solution

(a) $1-6x-x^2 = -x^2-6x+1$

$$= -\left[x^2+6x-1\right]$$

$$= -\left[x^2+6x+9-9-1\right]$$

$$\left(\frac{1}{2}\times\frac{6}{1}\right)^2 = (3)^2 = 9$$

$$= -\left[(x+3)^2-10\right]$$

$$= -(x+3)^2+10$$

$$= 10-(x+3)^2 \quad \left\{\text{in the form } k-a(x+h)^2\right\}$$

(b) i) The minimum value of $1-6x-x^2$ is 10.

ii) The equation of the axis of symmetry of $1-6x-x^2$ is $x=-\dfrac{(-6)}{2(-1)}=\dfrac{6}{-2}=-3$

So, the equation of the axis of symmetry of $1-6x-x^2$ is $x=-3$

(c) If $1-6x-x^2=0$, then $10-(x+3)^2=0$

Now, $-(x+3)^2=-10$

So, $(x+3)^2=10$

Now, $x+3=\pm\sqrt{10}$

So, $x=-3\pm\sqrt{10}$

Now, $x=-3+\sqrt{10}$ OR $x=-3-\sqrt{10}$

So, $x = -3 + 3.16$ OR $x = -3 - 3.16$

$\therefore$ $x = 0.16$ OR $x = -6.16$

Past paper (January 2009)

(a) Express the function $f(x) = 2x^2 - 4x - 13$ in the form $a(x+h)^2 + k$. (3 marks)

(b) Hence, or otherwise, determine

(i) the values of x at which the graph cuts the $x-$axis (4 marks)

(ii) the interval for which $f(x) \le 0$ (2 marks)

(iii) the minimum value of $f(x)$ (1 mark)

(iv) the value of x at which $f(x)$ is a minimum (2 marks)

Solution

(a) $f(x) = 2x^2 - 4x - 13$

$$= 2\left[x^2 - 2x - \frac{13}{2}\right]$$

$$= 2\left[\underbrace{x^2 - 2x + 1} - 1 - \frac{13}{2}\right] \qquad \boxed{\left(\frac{1}{2} \times \frac{-2}{1}\right)^2 = (-1)^2 = 1}$$

$$= 2\left[(x-1)^2 - \frac{2}{2} - \frac{13}{2}\right]$$

$$= 2\left[(x-1)^2 - \frac{15}{2}\right]$$

$f(x) = 2(x-1)^2 - 15$ $\quad \{\text{in the form } a(x+h)^2 + k\}$

(b) (i) At the points where the graph cuts the $x-$axis, $y = 0$.

Substituting $y = 0$ into the equation $y = 2x^2 - 4x - 13$ gives

$$2x^2 - 4x - 13 = 0$$

Now, $2x^2 - 4x - 13 = 2(x-1)^2 - 15$

So, $2(x-1)^2 - 15 = 0$

Now, $2(x-1)^2 = 15$

So, $(x-1)^2 = \frac{15}{2}$

Now, $x-1 = \sqrt{\frac{15}{2}}$

So, $x-1 = \pm 2.74$

Now, $x-1 = -2.74$ OR $x-1 = 2.74$

So, $x = -2.74+1$ OR $x = 2.74+1$

$\therefore$ $x = -1.74$ OR $x = 3.74$

(b) (ii) $f(x) \le 0$ when $-1.74 \le x \le 3.74$

(b) (iii) $f(x) = 2(x-1)^2 - 15$ $\{$in the form $a(x+h)^2 + k\}$

The minimum value of $f(x)$ is k

So, the minimum value of $f(x)$ is -15

(b) (iv) The minimum point on $f(x)$ is the point $(-h, k) = (1, -15)$

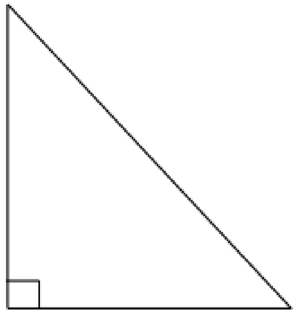
So, the value of x at which $f(x)$ is a minimum is 1

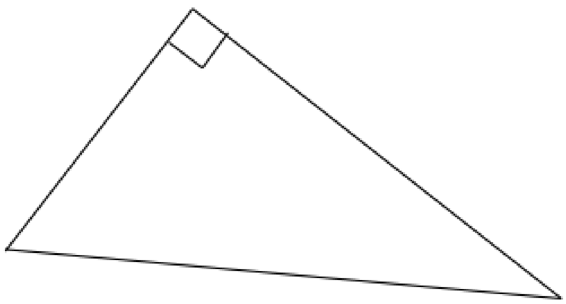
TRIGONOMETRY

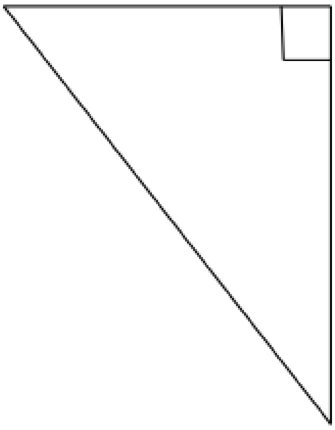
Pythagoras' theorem is associated with ONLY right angled triangles.

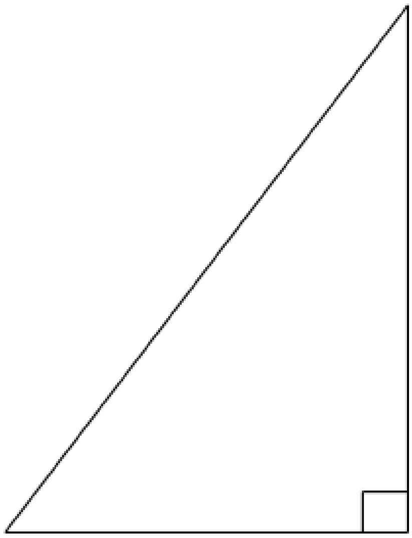
The longest side of a right angled triangle is called the HYPOTHENUSE.

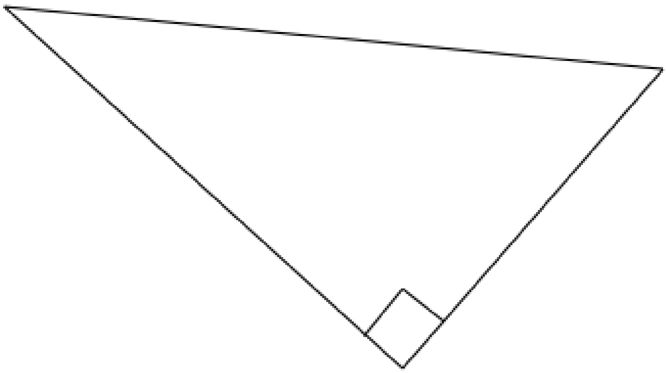
The hypothenuse of a right angled triangle may be identified irrespective of the orientation of the triangle.



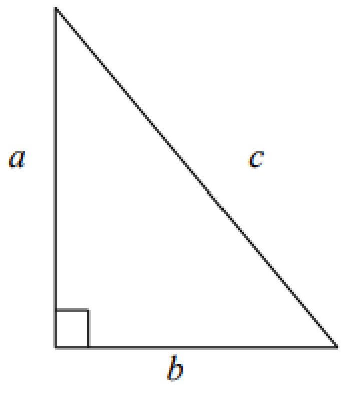








Pythagoras' theorem states that the square of the hypothenuse is equal to the sum of the squares of the other two sides.



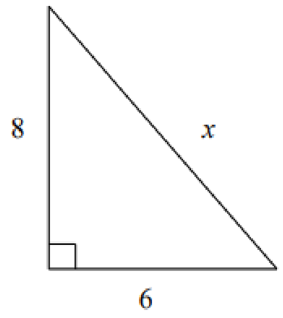
Let a represent the perpendicular side of a right angled triangle, b represents its base and c represents its hypothenuse.

Now, $\text{hypothenuse}^2 = \text{perpendicular}^2 + \text{base}^2$

So, $c^2 = a^2 + b^2$

So, $c = \sqrt{a^2 + b^2}$

Note: If we know the lengths of two sides of a right angled triangle, we may use Pythagoras' theorem to find the third side.



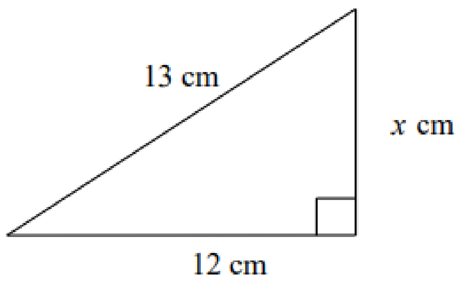
Here, $x^2 = 8^2 + 6^2$

So, $x^2 = 64 + 36$

So, $x^2 = 100$

So, $x = \sqrt{100}$

Therefore $x = 10$



Here, $13^2 = 12^2 + x^2$

So, $169 = 144 + x^2$

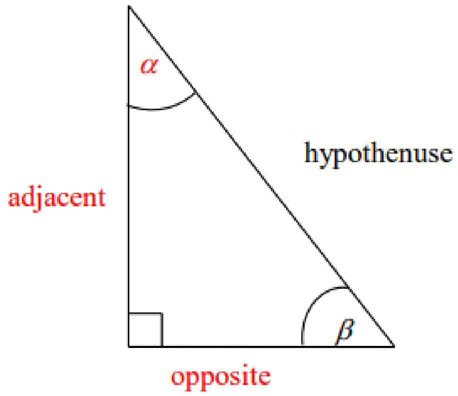
So, $169 - 144 = x^2$

So, $x^2 = 25$

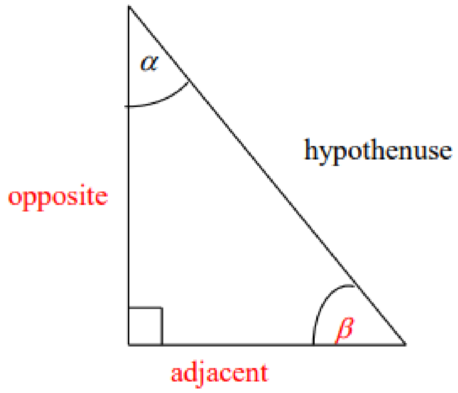
So, $x = \sqrt{25}$

Therefore $x = 5$ cm

Trigonometric ratios of an angle in a right angled triangle



In relation to the angle α, the side opposite to α is labelled 'opposite' and the side which is perpendicular to the opposite side is labelled 'adjacent'.

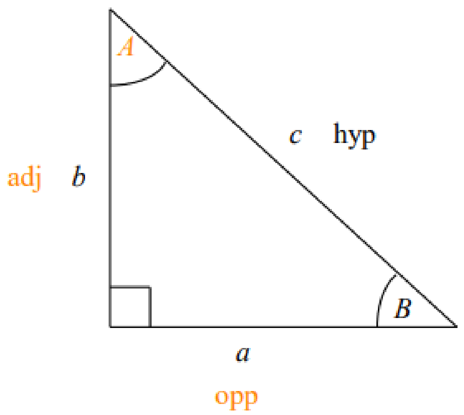


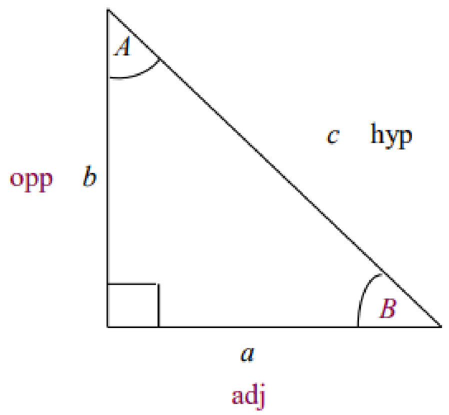
In relation to the angle β, the side opposite to β is labelled 'opposite' and the side which is perpendicular to the opposite side is labelled 'adjacent'.

If θ is an interior angle of a right angled triangle, then

$$\sin\theta = \frac{\text{opposite}}{\text{hypothenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypothenuse}} \quad \text{and} \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

SOH CAH TOA





Now, $\sin A = \dfrac{a}{c}$

Also, $\cos A = \dfrac{b}{c}$

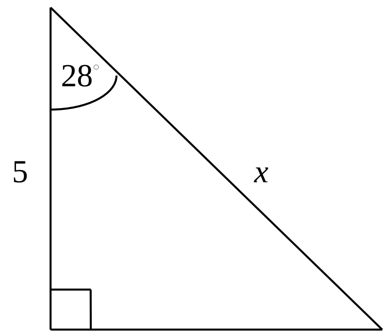
Also, $\tan A = \dfrac{a}{b}$

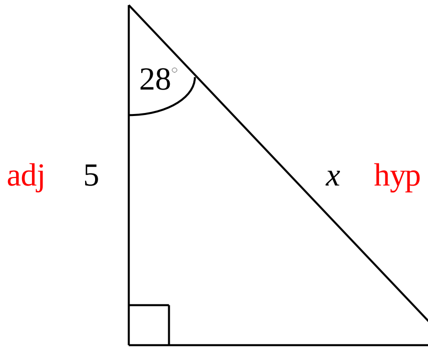
Now, $\sin B = \dfrac{b}{c}$

Also, $\cos B = \dfrac{a}{c}$

Also, $\tan B = \dfrac{b}{a}$

Finding unknown sides and angles using trigonometric ratios





Here, $\frac{\cos 28^\circ}{1} = \frac{5}{x}$

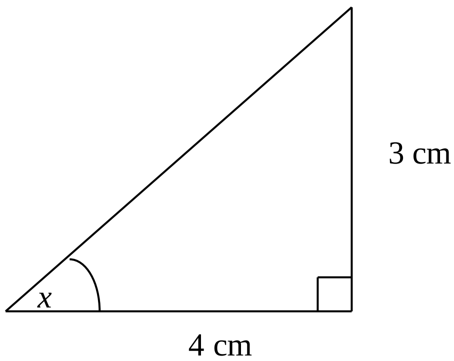
So, $x \times \cos 28^\circ = 5$

Then $x = \frac{5}{\cos 28^\circ}$

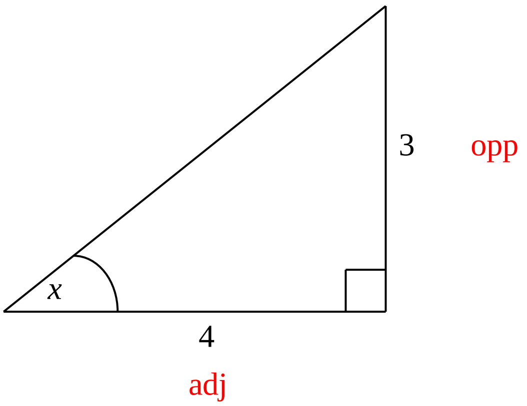
So, $x = 5 \cdot 66$

Example

Find x



Solution



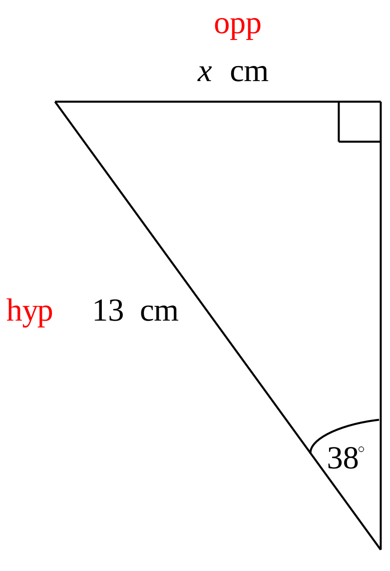
Here, $\tan x = \frac{3}{4}$

So, $x = \tan^{-1}\frac{3}{4} \quad = \tan^{-1} 0\cdot75$

So, $x = 36\cdot9^{\circ}$

Example

Find x



Solution

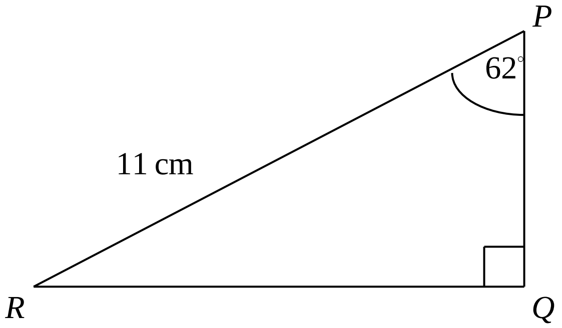
$\frac{\sin 38^{\circ}}{1} = \frac{x}{13}$

So, $x = 13 \times \sin 38^{\circ}$

Then $x = 8$ cm

Example (May 2021)

The diagram below shows the triangle PQR in which angle $QPR = 62^\circ$, angle $PQR = 90^\circ$ and $PR = 11$ cm.



Calculate

(a) the size of angle PRQ (1 mark)

(b) the length of the side RQ (2 marks)

Solution

(a) angle $PRQ = 180^\circ - (90^\circ + 62^\circ)$

$= 180^\circ - 152^\circ$

$= 28^\circ$

(b) $\sin 62^\circ = \frac{RQ}{11}$

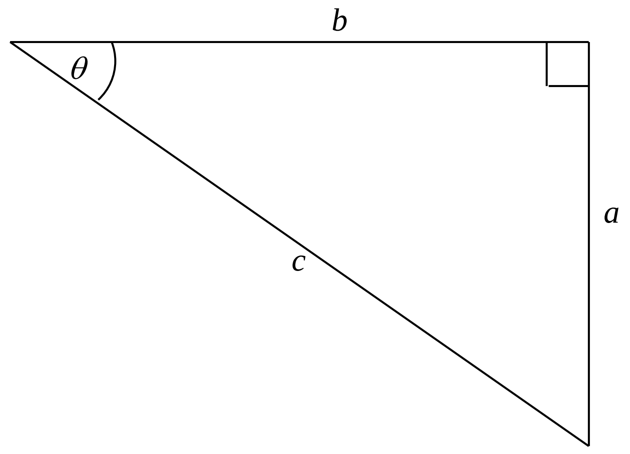
Now, $\frac{\sin 62^\circ}{1} = \frac{RQ}{11}$

Now, $RQ = 11 \times \sin 62^\circ$

Now, $RQ = 9 \cdot 71$ cm

Example (January 2019)

The diagram below shows a right-angled triangle with sides a units, b units and c units.



(a) Using the diagram,

(i) express c in terms of a and b (1 mark)

(ii) write, in terms of a, b and c, an expression for $\sin\theta + \cos\theta$. (2 marks)

(b) Using the results in (a) (i) and (ii) above, show that $(\sin\theta)^2 + (\cos\theta)^2 = 1$

(2 marks)

Solution

(a) (i) Using Pythagoras' theorem, $c^2 = a^2 + b^2$

(a) (ii) From the triangle, $\sin\theta = \frac{a}{c}$

From the triangle, $\cos\theta = \frac{b}{c}$

So, $\sin\theta + \cos\theta = \frac{a}{c} + \frac{b}{c}$

(b) Now, $(\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$

$$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

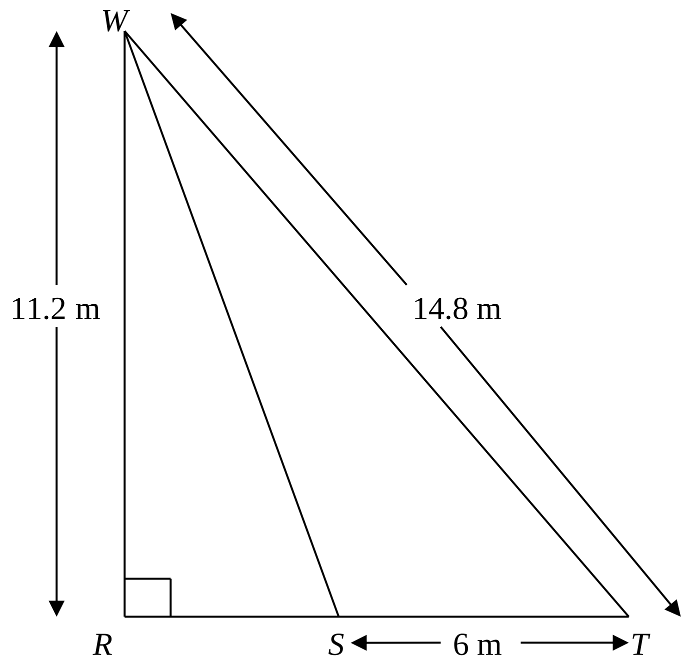
$$= \frac{a^2 + b^2}{c^2}$$

$$= \frac{c^2}{c^2} \text{ , since } c^2 = a^2 + b^2$$

$$= 1$$

Example (January 2016)

In the diagram below, not drawn to scale, $\text{ST} = 6\,\text{m}$, $\text{WR} = 11.2$ m, $\text{WT} = 14.8$ m and and angle $\text{WRS} = 90^\circ$.



Calculate, giving your answer to 1 decimal place

(a) the length RS (2 marks)

(b) the measure of angle RTW (2 marks)

Solution

(a) By considering triangle WRT, $WT^2 = WR^2 + RT^2$

Now, $14.8^2 = 11.2^2 + RT^2$

So, $RT^2 = 14.8^2 - 11.2^2$

Now, $RT^2 = 219.0 - 125.4$

So, $RT^2 = 93.6$

Now, $RT = \sqrt{93.6}$

So, $RT = 9.7$ m

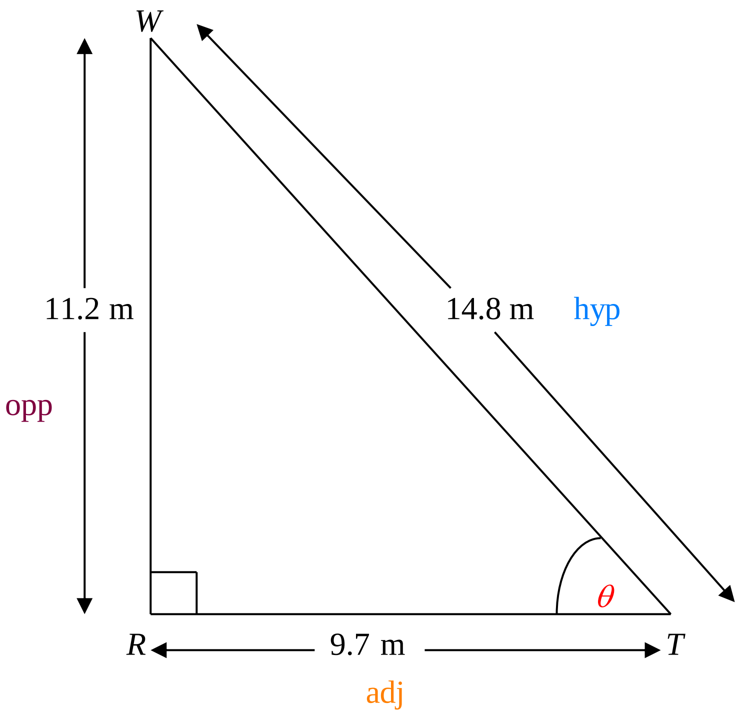
Now, $RT = RS + ST$

So, $RS = RT - ST$

Now, $RS = 9.7 - 6$

$\therefore \quad RS = 3.7 \text{ m}$

(b)



Let angle $RTW = \theta$

Now, $\sin\theta = \dfrac{11.2}{14.8}$

So, $\sin\theta = 0.757$

Now, $\theta = \sin^{-1} 0.757$

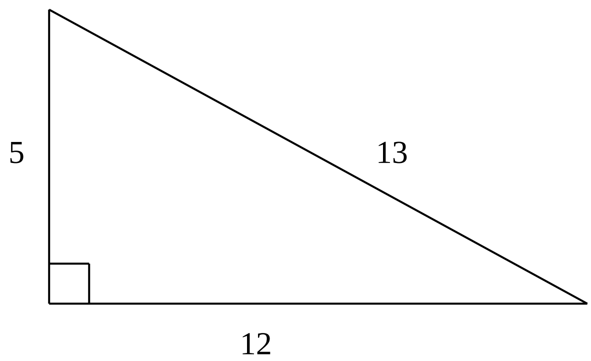
$\therefore \quad \theta = 49.2^\circ$

So, angle $RTW = 49.2^\circ$

Area of a triangle

It is easy to find the area of a right angled triangle. Here, we use the formula 'half base times perpendicular height'.

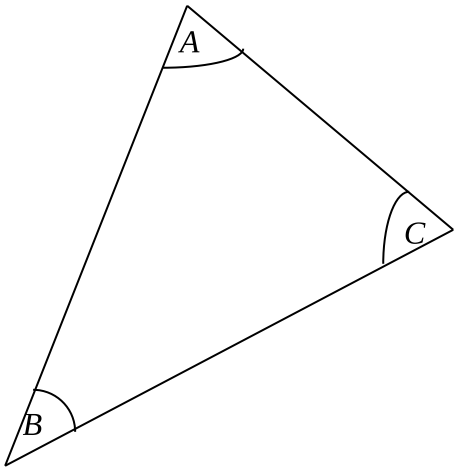
Consider the right angled triangle shown below:



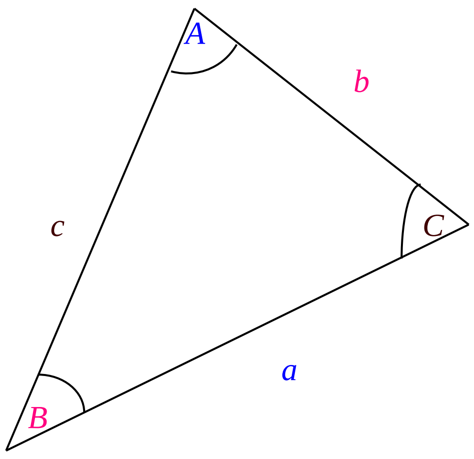
Here, the area of the triangle is $\frac{1}{2}(12)\times 5 = 6\times 5 = 30$ squared units

Now, a different approach has to be taken to find the area of a triangle which is NOT a right angled triangle.

Consider a triangle with the interior angles A, B and C as shown below



The sides OPPOSITE to angles A, B and C are a, b and c respectively.



Now, the area of a triangle may be found if the lengths of two of its sides and the angle between these two sides are known.

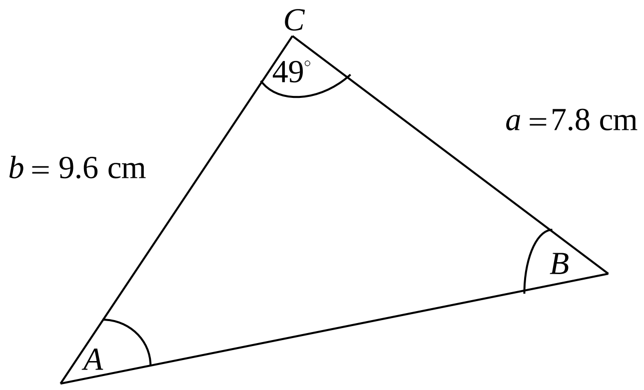
So, if we know sides a and b and angle C, we may find the area of the triangle.
Similarly, if we know sides a and c and angle B, we may find the area of the triangle.
Similarly, if we know sides b and c and angle A, we may find the area of the triangle.

Now, if we know sides a and b and angle C, the area, A, of the triangle is given by $A=\frac{1}{2}ab\sin C$

Similarly, $A=\frac{1}{2}ac\sin B$

Also, $A=\frac{1}{2}bc\sin A$

Consider the triangle shown below



Here, the area of the triangle, A, is given by $A=\frac{1}{2}ab\sin C$

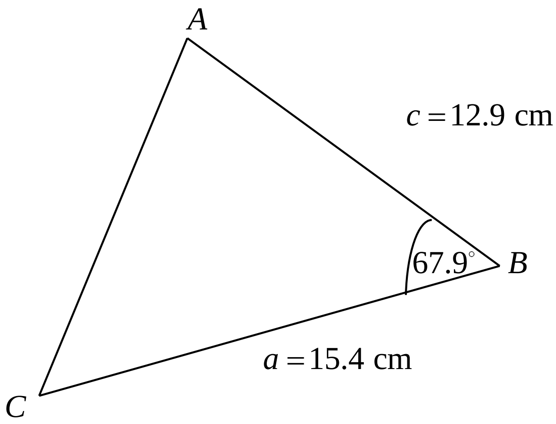
So, $A=\frac{1}{2}\times(7.8)\times(9.6)\times\sin 49^{\circ}$

So, $A = 28.3\ \text{cm}^2$

Example

ABC is a triangle in which $AB = 12.9$ cm, $CB = 15.4$ cm and angle $ABC = 67.9^{\circ}$, calculate the area of triangle ABC.

Solution



Now, the area, A, of triangle $ABC = \frac{1}{2}ac\sin B$

So, $A=\frac{1}{2}(15.4)(12.9)\sin 67.9^{\circ}$

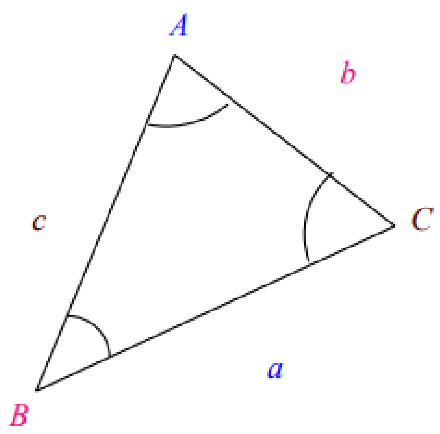
$\therefore\quad A = 92.0\ \text{cm}^2$

Sine Rule

The sine rule is used to find unknown sides and angles of a triangle which is not a right angled triangle.

The sine rule may be applied if either of the following is known:
(i) two angles and the length of one side of a triangle
(ii) two sides and an angle opposite to one of the two sides

Consider the triangle shown below



Now, the SINE RULE states that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

It implies that $\frac{a}{\sin A} = \frac{b}{\sin B}$

Also, $\frac{a}{\sin A} = \frac{c}{\sin C}$

Also, $\frac{b}{\sin B} = \frac{c}{\sin C}$

Example

In ΔABC, $a = 18$ cm, $\sin A = 0.6$ and $b = 8$ cm. Calculate $\sin B$

Solution

Here, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

So, $\dfrac{18}{0.6} = \dfrac{8}{\sin B}$

Now, $18 \times \sin B = 8 \times 0.6$

So, $18 \times \sin B = 4.8$

So, $\sin B = \dfrac{4.8}{18}$

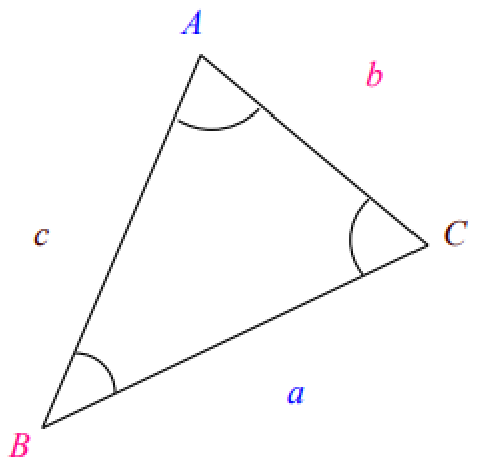
$\therefore$ $\sin B = 0.267$

Cosine Rule

The cosine rule is used to find unknown sides and angles of a triangle which is not a right angled triangle.

The cosine rule may be applied if either of the following is known:
(i) the length of two sides of a triangle and the angle between both sides
(ii) the lengths of the three sides of a triangle

Consider the triangle shown below



Now, the COSINE RULE states that $a^2 = b^2 + c^2 - 2bc\cos A$

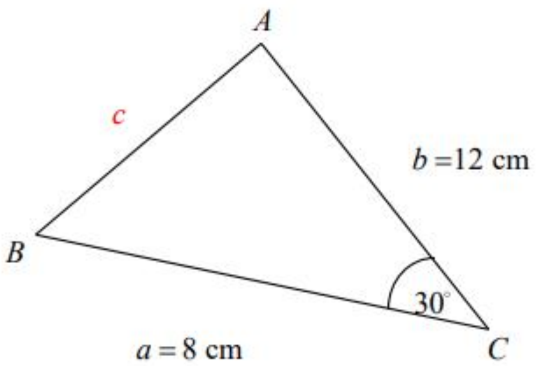
It implies that $b^2 = a^2 + c^2 - 2ac\cos B$

Also, $c^2 = a^2 + b^2 - 2ab\cos C$

Example

In ΔABC, $a = 8$ cm, $b = 12$ cm and $\hat{C} = 30^\circ$. Determine the length of side c.

Solution



Here, $c^2 = a^2 + b^2 - 2ab\cos C$

So, $c^2 = (8)^2 + (12)^2 - 2(8)(12)\cos 30^\circ$

So, $c^2 = 64 + 144 - 2(8)(12)\cos 30^\circ$ So, $c^2 = 208 - 192\cos 30^\circ$

So, $c^2 = 208 - 166.28$

So, $c^2 = 41.72$

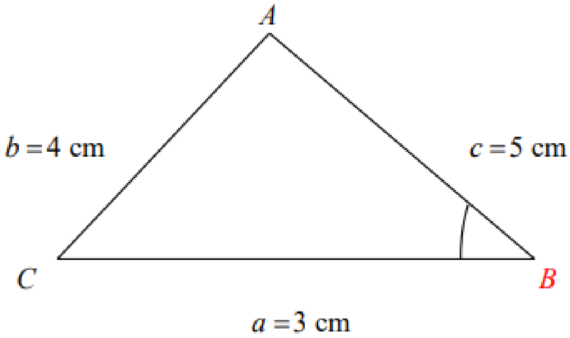
Now, $c = \sqrt{41.72}$

$\therefore$ $c = 6.46$ cm

Example

If $a=3$ cm, $b=4$ cm and $c=5$ cm in ΔABC, determine the magnitude of angle B.

Solution



Here, $b^2 = a^2 + c^2 - 2ac\cos B$

So, $(4)^2 = (3)^2 + (5)^2 - 2(3)(5)\cos B$

Now, $16 = 9 + 25 - 30\cos B$

So, $16 = 34 - 30\cos B$ Now, $16 - 34 = 34 - 30\cos B - 34$

Now, $-30\cos B = -18$

So, $\cos B = \dfrac{-18}{-30}$

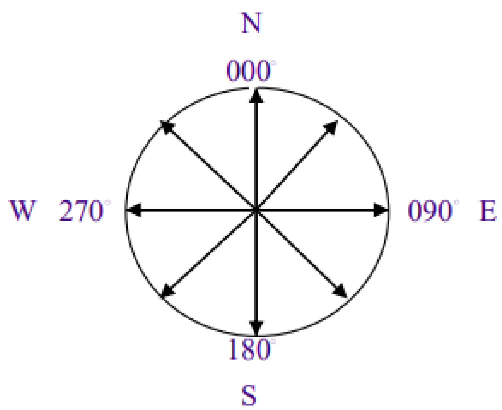
Now, $\cos B = 0.6$

So, $B = \cos^{-1} 0.6$

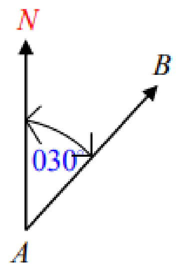
$\therefore B = 53.1^\circ$

BEARINGS

The position of an object relative to another object is called its BEARING.
The bearing of an object is the ANGLE measured in a CLOCKWISE direction from NORTH to the object.
Bearings are always written using three digits in the whole number part.

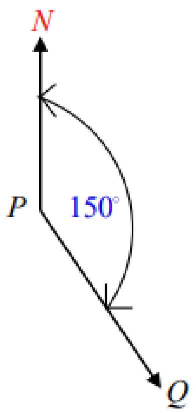


Consider the sketch below:



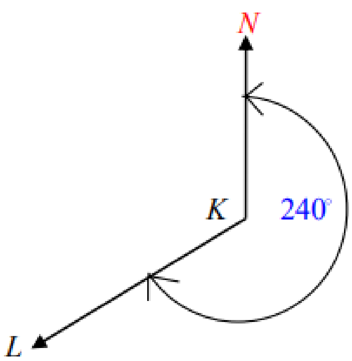
The bearing of B from A is 030°

Consider the sketch below:



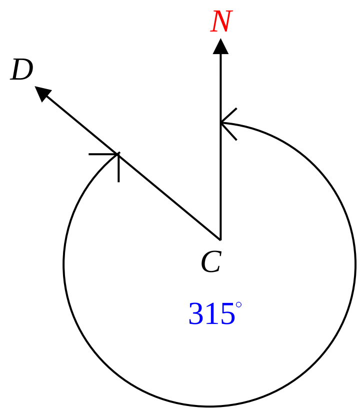
The bearing of Q from P is 150°

Consider the sketch below:



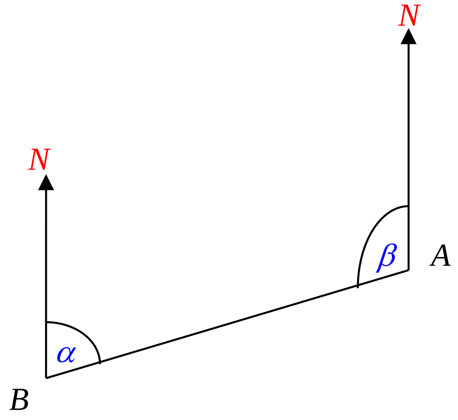
The bearing of L from K is 240°

Consider the sketch below:



The bearing of D from C is 315°

Note:

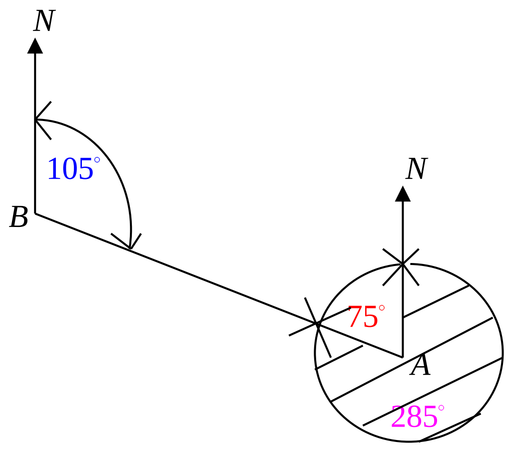


Since NB is parallel to NA, then $\alpha + \beta = 180^{\circ}$ (supplementary angles)

Example

The bearing of a point A from a point B is 105°. State the bearing of B from A.

Solution



Since NB and NA are parallel, then angle $NAB = 180^\circ - 105^\circ = 75^\circ$

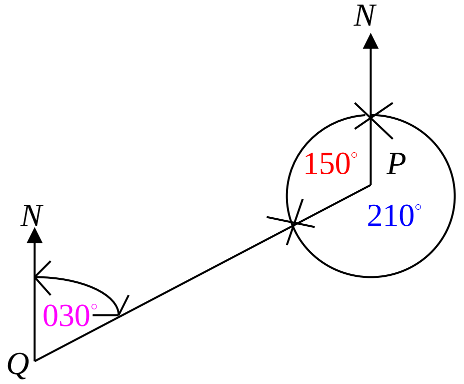
Now, $360^\circ - 75^\circ = 285^\circ$

So, the bearing of B from A is 285°

Example

The bearing of a point Q from a point P is 210°. What is the bearing of P from Q?

Solution



Angle $NPQ = 360^\circ - 210^\circ = 150^\circ$

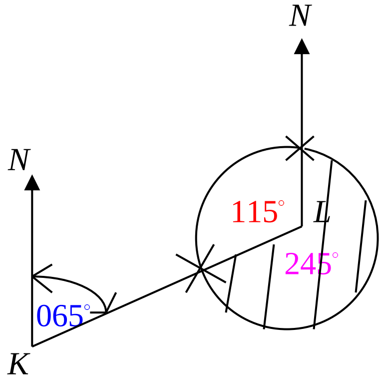
Since NP and NQ are parallel, angle $NQP = 180^\circ - 150^\circ = 030^\circ$

$\therefore$ the bearing of P from Q is 030°

Example

The bearing of a boat L from a harbour K is 065°. What is the bearing of the harbour K from the boat L?

Solution



Since NK and NL are parallel, then angle $NLK = 180^{\circ} - 065^{\circ} = 115^{\circ}$

Now, $360^{\circ} - 115^{\circ} = 245^{\circ}$

So, the bearing of the harbour K from the boat L is 245°

Example (January 2007)

A boat leaves a dock at point A and travels for a distance of 15 km to a point B on a bearing of 135°.

The boat then changes course and travels for a distance of 8 km to a point C on a bearing of 060°.

(a) Illustrate the above information in a clearly labeled diagram (2 marks)

The diagram should show the

(i) north direction (1 mark)

(ii) bearings 135° and 060° (2 marks)

(iii) distances 8 km and 15 km (2 marks)

(b) Calculate

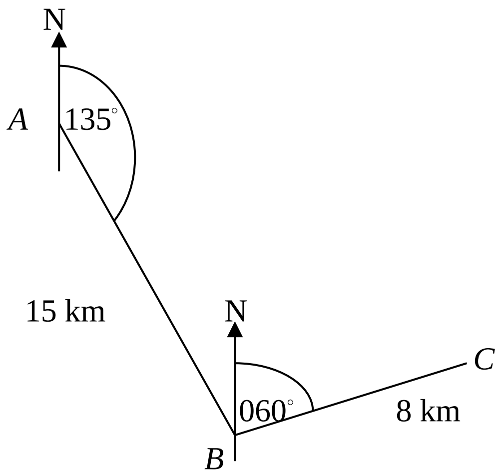
(i) the distance AC (3 marks)

(ii) $\angle BCA$ (3 marks)

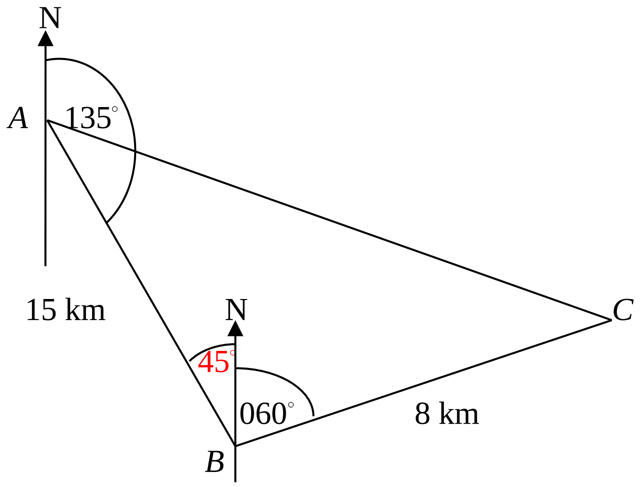
(iii) the bearing of A from C (2 marks)

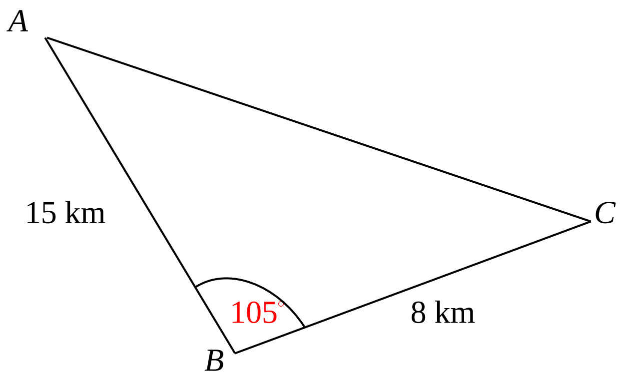
Solution

(a)



(b) (i)





Now, $AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos B$

So, $AC^2 = 15^2 + 8^2 - 2(15)(8)\cos 105^\circ$

So, $AC^2 = 225 + 64 - 240\cos 105^\circ$

Now, $AC^2 = 289 - 240\cos 105^\circ$

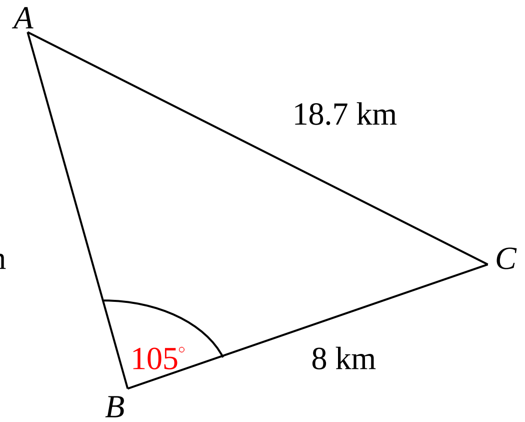
So, $AC^2 = 289 - (-62.1)$

So, $AC^2 = 351.1$

Now, $AC = \sqrt{351.1}$

$\therefore \quad AC = 18.7 \text{ km}$

(b) (ii)



Now, $\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C}$

So, $\dfrac{18.7}{\sin 105^\circ} = \dfrac{15}{\sin C}$

Now, $18.7 \times \sin C = 15 \times \sin 105^\circ$

Now, $\sin C = \dfrac{15 \times \sin 105^\circ}{18.7}$

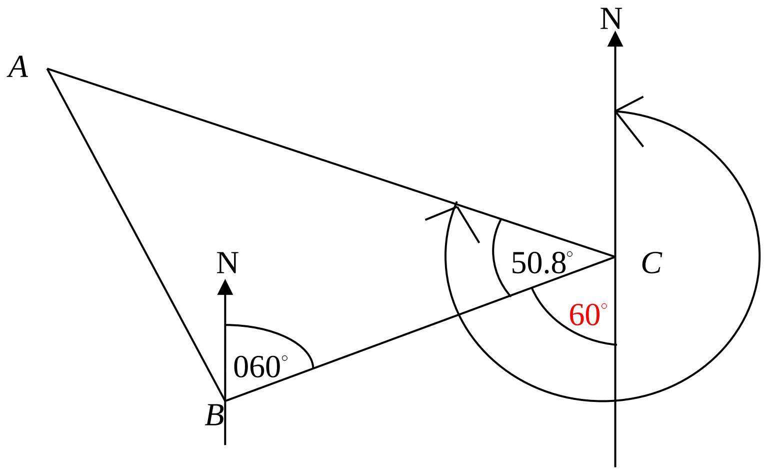
So, $\sin C = 0.775$

So, $C = \sin^{-1} 0.775$

So, $C = 50.8^\circ$

$\therefore \quad \angle BCA = 50.8^\circ$

(b) (iii)



Now, $180^\circ + 60^\circ + 50.8^\circ = 290.8^\circ$

So, the bearing of A from C is 290.8°

Example

(a) Sketch a diagram to represent the information given below.
Show clearly all measurements and any north-south lines that may be required.

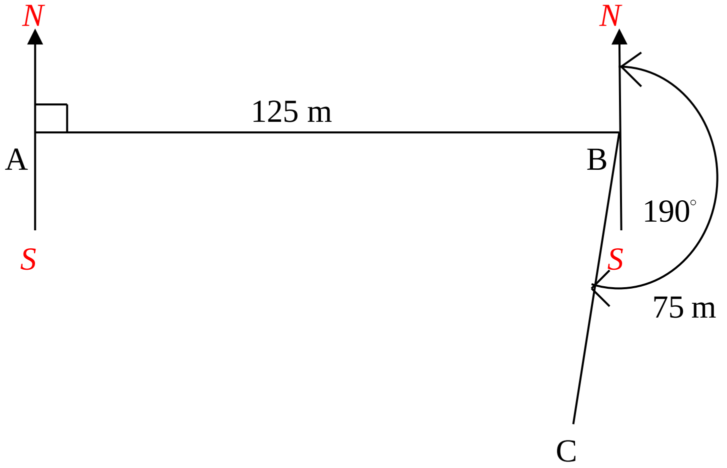
A, B and C are three buoys.
B is 125 m due east of A.
The bearing of C from B is 190°.
$CB = 75$ m. (5 marks)

(b) Calculate, to one decimal place, the distance AC. (3 marks)

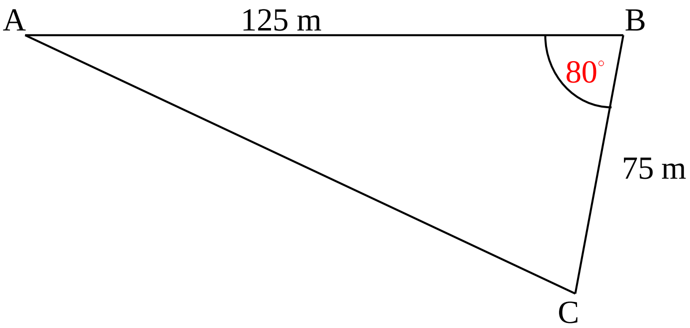
(c) Calculate, to the nearest degree, the bearing of C from A. (3 marks)

Solution

(a)



(b)



Now, $AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos B$

So, $AC^2 = 125^2 + 75^2 - 2(125)(75)\cos 80^\circ$

Now, $AC^2 = 15625 + 5625 - 18750\cos 80^\circ$

So, $AC^2 = 21250 - 18750\cos 80^\circ$

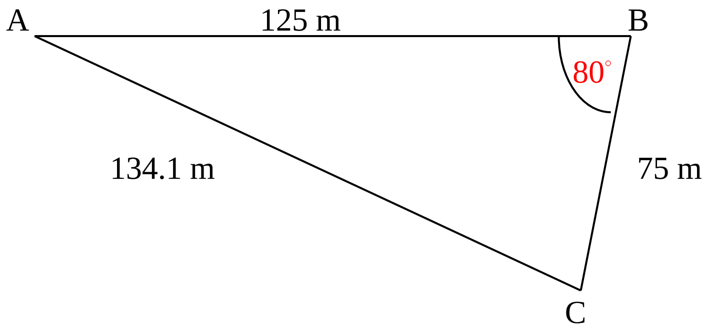
Now, $AC^2 = 21250 - 3255.9$

So, $AC^2 = 17994.1$

Now, $AC = \sqrt{17994.1}$

$\therefore \quad AC = 134.1 \text{ m}$

(c) Here, we need $\angle BAC$



Here, $\dfrac{BC}{\sin A} = \dfrac{AC}{\sin B}$

So, $\dfrac{75}{\sin A} = \dfrac{134.1}{\sin 80^\circ}$

Now, $134.1 \times \sin A = 75 \times \sin 80^\circ$ Now, $\sin A = \dfrac{75 \times \sin 80^\circ}{134.1} = 0 \cdot 551$

So, $A = \sin^{-1} 0 \cdot 551$

$\therefore$ $A = 33^\circ$ (to the nearest degree)

(c) Now, $90^\circ + 33^\circ = 123^\circ$

$\therefore$ the bearing of C from A is 123°

<u>Example (May 2008)</u>

A ship leaves Port R, sails to Port S and then to Port T.

The bearing of S from R is 112° ; The bearing of T from S is 033°.

The distance RT is 75 km and the distance RS is 56 km.

(a) Draw a diagram showing the journey of the ship from R to S to T.
Show on your diagram

(i) the North direction (1 mark)

(ii) the bearings 112° and 033° (2 marks)

(iii) the points R, S and T (1 mark)

(iv) the distances 75 km and 56 km (1 mark)

(b) Calculate

(i) the size of angle RST (1 mark)

(ii) the size of angle RTS (3 marks)

(iii) the bearing of R from T (2 marks)

(c) The ship leaves Port T and travels due west to a point X which is due north of R.

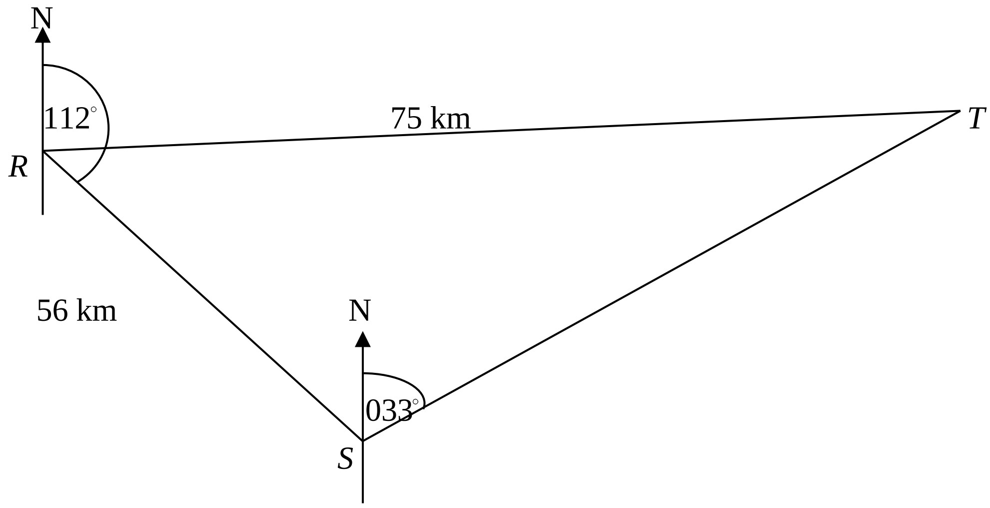
(i) Show on your diagram the journey from T to X (1 mark)

(ii) Calculate the distance TX (3 marks)

Total 15 marks

Solution

(a)



(b) (i) Now, angle $RST = (180^\circ - 112^\circ) + 33^\circ$

$= 68^\circ + 33^\circ$

$\therefore$ angle $RST = 101^\circ$

(b) (ii) Now, $\dfrac{RS}{\sin T} = \dfrac{RT}{\sin S}$

So, $\dfrac{56}{\sin T} = \dfrac{75}{\sin 101^\circ}$ Now, $56 \times \sin 101^\circ = 75 \times \sin T$

Now, $\sin T = \dfrac{56 \times \sin 101^\circ}{75} = 0 \cdot 733$

So, $T = \sin^{-1} 0 \cdot 733$

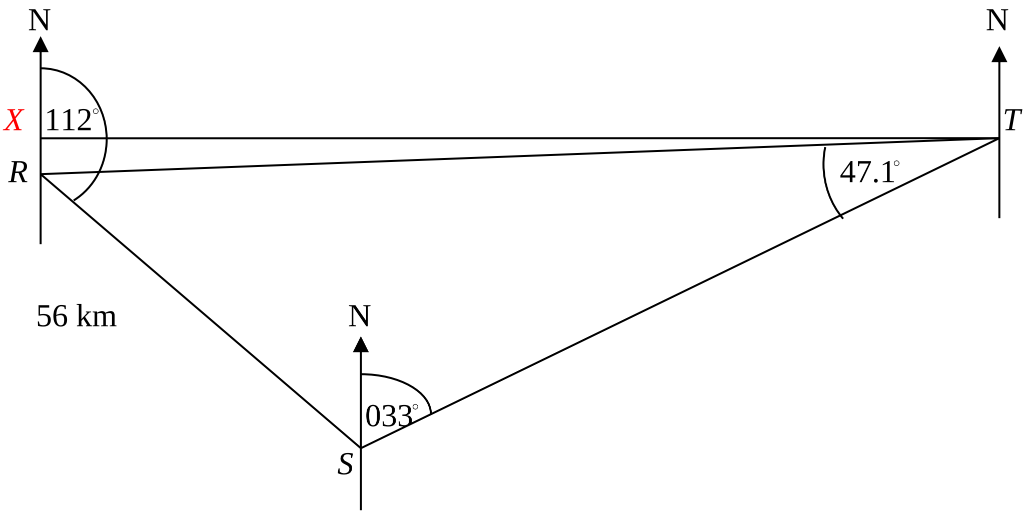
So, $T = 47.1^\circ$

$\therefore$ angle $RTS = 47.1^\circ$

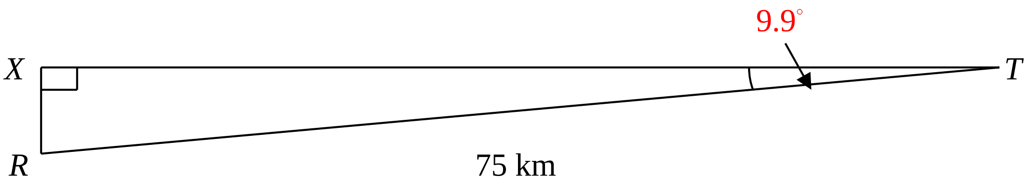
(b) (iii) Now, $180^\circ + 33^\circ + 47.1^\circ = 260.1^\circ$

$\therefore$ the bearing of R from T is 260.1°

(c) (i)



(c) (ii)



$\angle XTR = 270^\circ - 260.1^\circ$

So, $\angle XTR = 9.9^\circ$

Now, $\frac{\cos 9.9^\circ}{1} = \frac{TX}{75}$

So, $TX = 75 \times \cos 9.9^\circ$

$\therefore\ TX = 73.9$ km

Example (January 2011)

J, K and L are three sea ports. A ship began its journey at J, sailed to K, then to L and returned to J.

The bearing of K from J is 054° and L is due east of K.
$JK = 122$ km and $KL = 60$ km

(a) Draw a clearly labelled diagram to represent the above information. Show on the diagram

i) the north/south direction

ii) the bearing 054°

iii) the distances 122 km and 60 km (3 marks)

(b) Calculate

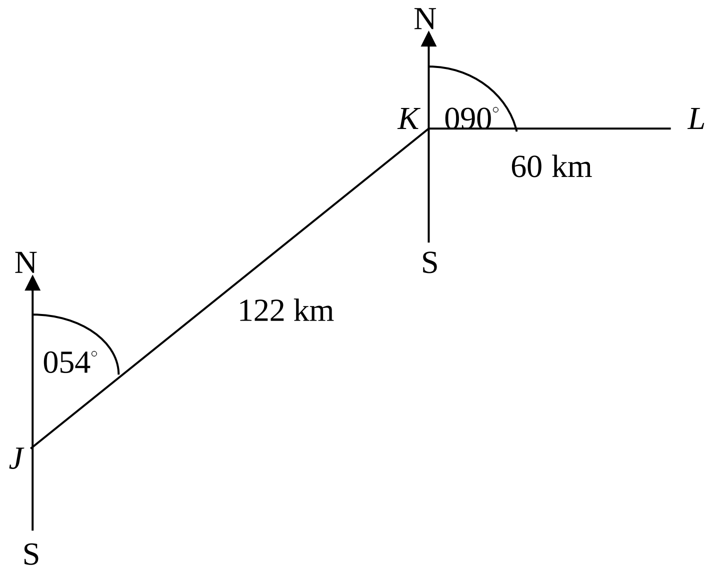
i) the measure of angle JKL

ii) distance JL

iii) the bearing of J from L. (7 marks)

Solution

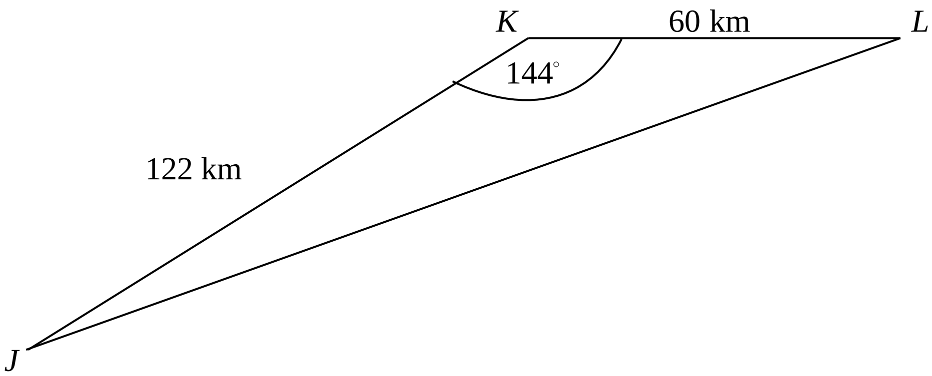
(a)



(b) (i) Angle $JKL = 54^\circ + 90^\circ$

$= 144^\circ$

(b) (ii)



Now, $JL^2 = JK^2 + KL^2 - 2(JK)(KL)\cos K$

So, $JL^2 = 122^2 + 60^2 - 2(122)(60)\cos 144^\circ$

So, $JL^2 = 14884 + 3600 - 14640\cos 144^\circ$

Now, $JL^2 = 18484 - (-11844)$

So, $JL^2 = 30328$

Now, $JL = \sqrt{30328}$

$\therefore \quad JL = 174 \text{ km}$

(b) (iii) Here, we need to find angle KLJ.

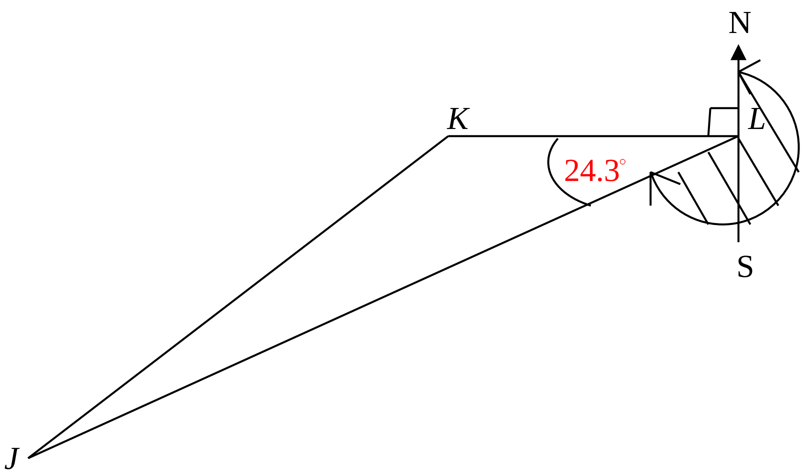
Now, $\dfrac{JL}{\sin K}=\dfrac{JK}{\sin L}$

So, $\dfrac{174}{\sin 144^\circ}=\dfrac{122}{\sin L}$

Now, $174\times\sin L=122\times\sin 144^\circ$ Now, $\sin L=\dfrac{122\times\sin 144^\circ}{174}=0\cdot 412$

So, $L=\sin^{-1}0\cdot 412$

$\therefore\ L=24.3^\circ$

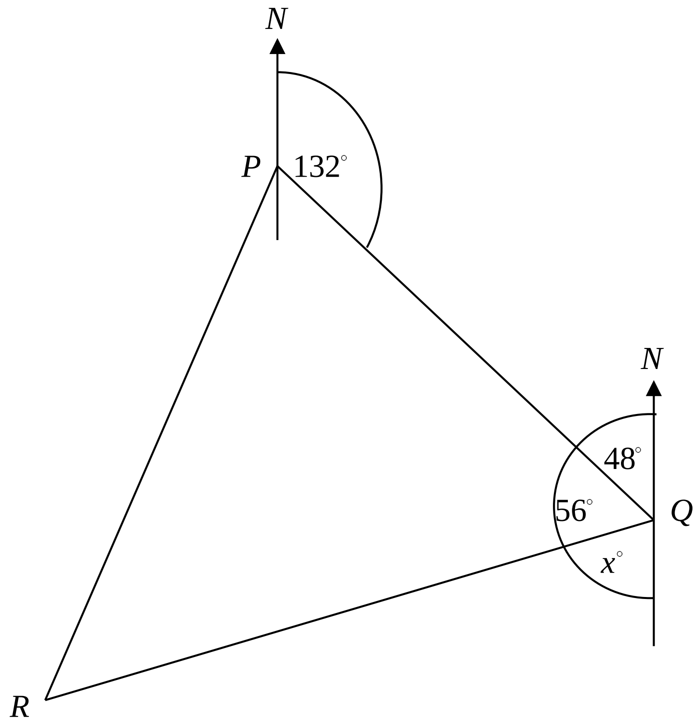


Now, $270^\circ-24.3^\circ=245.7^\circ$

$\therefore$ the bearing of J from L is 245.7°

TRY THIS (May 2011)

4. The diagram below, not drawn to scale, shows the route of an aeroplane flying from Portcity (P) to Queenstown (Q) and then to Riversdale (R). The bearing of Q from P is 132° and the angle PQR is 56°.



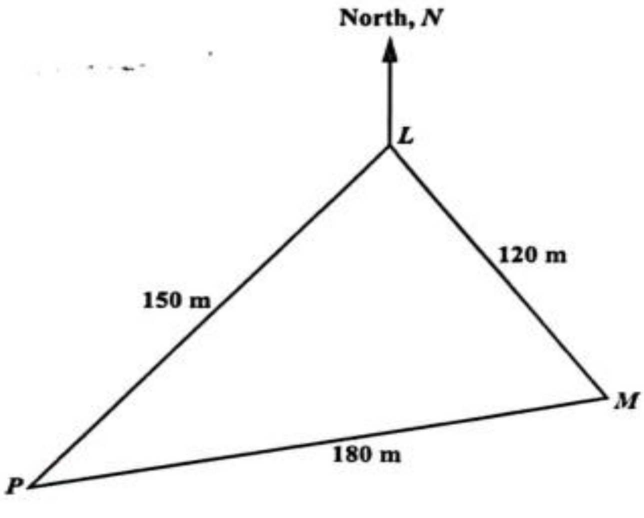
(a) Calculate the value of x, as shown in the diagram (1 mark)

(b) The distance from Portcity (P) to Queenstown (Q) is 220 kilometres and the distance from Queenstown to Riversdale (R) is 360 kilometres. Calculate the distance RP (3 marks)

(c) Determine the bearing of R from P (3 marks)

TRY THIS (May 2023)

5. The diagram below shows a triangular field, LMP, on horizontal ground.



(a) Calculate the value of Angle MLP (3 marks)

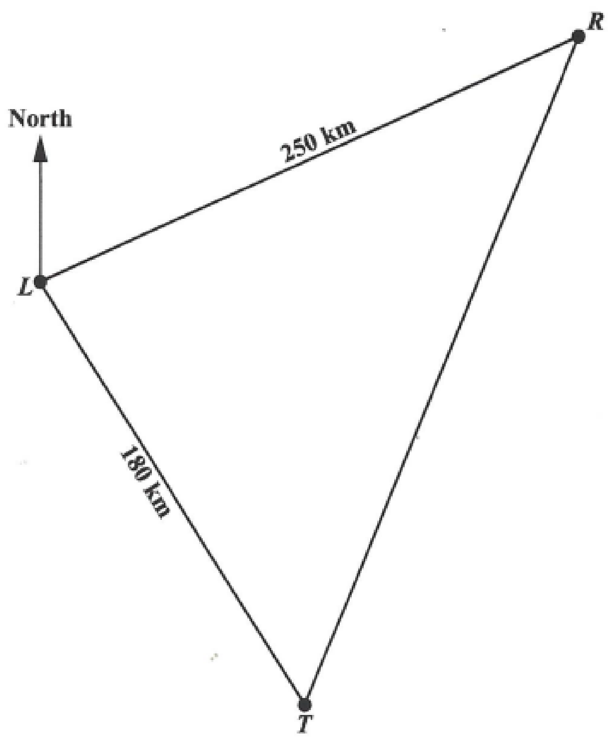
(b) The bearing of P from L is 210°

i) Find the bearing of M from L (1 mark)

ii) Calculate the value of Angle NLP and hence, find the bearing of L from P (2 marks)

TRY THIS (May 2022)

6. From a port, L, ship R is 250 kilometres on a bearing of 065°. Ship T is 180 kilometres from L on a bearing of 148°. This information is illustrated in the diagram below.



(a) Complete the diagram above by inserting the value of angle RLT (1 mark)

(b) Calculate RT, the distance between the two ships (2 marks)

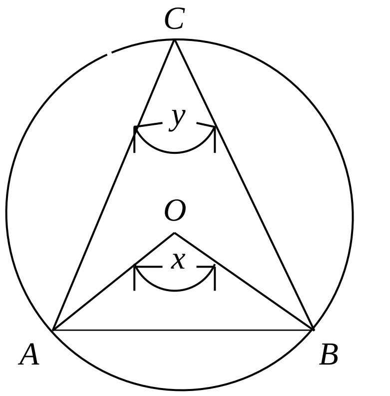
(c) Determine the bearing of T from R (3 marks)

CIRCLE THEOREM

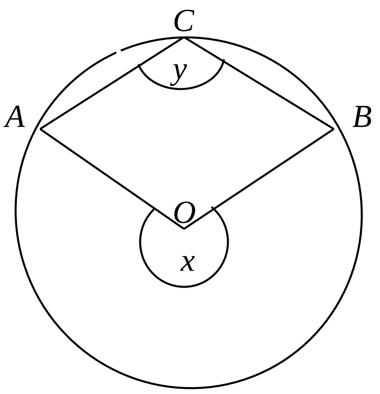
Circle Theorem

Theorem 1

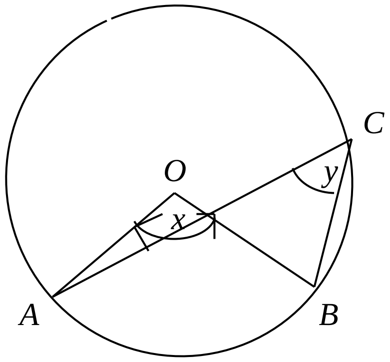
The angle at the centre of a circle is twice the angle at the circumference standing on the same arc (or chord).



$x = 2y$



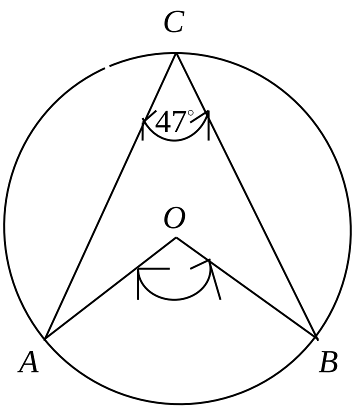
$x = 2y$



$x = 2y$

Example

If $A\hat{C}B = 47^\circ$, calculate $A\hat{O}B$, stating a reason for your answer.



Solution

$A\hat{O}B = 2A\hat{C}B$

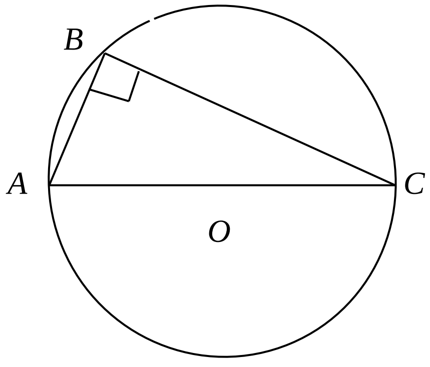
$= 2(47^\circ)$

$= 94^\circ$

The angle formed at the centre of a circle by an arc (or a chord) is twice the angle formed at the circumference by that arc (or chord).

Theorem 2

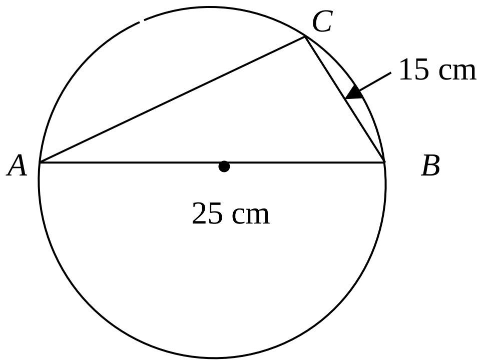
The angle formed by two chords within a semi-circle is a right angle.



$A\hat{B}C = 90^\circ$

Example

If AB is a diameter, evaluate AC, stating reasons for your answer.



Solution

Angle $ACB = 90^\circ$ since the chords AC and BC meet within a semi-circle.

So, ΔABC is right angled.

Now, $AB^2 = AC^2 + BC^2$

So, $25^2 = AC^2 + 15^2$

Now, $AC^2 = 25^2 - 15^2$

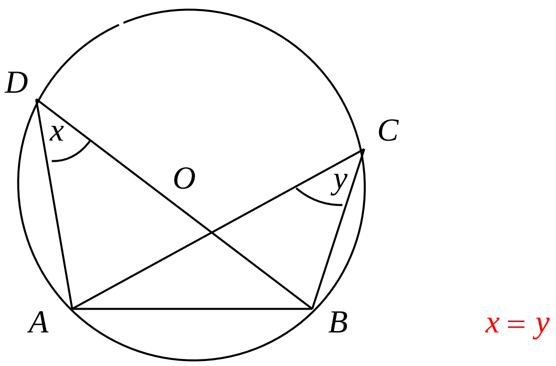
Then $AC^2 = 625 - 225 = 400$

Now, $AC = \sqrt{400}$

$\therefore$ $AC = 20$ cm

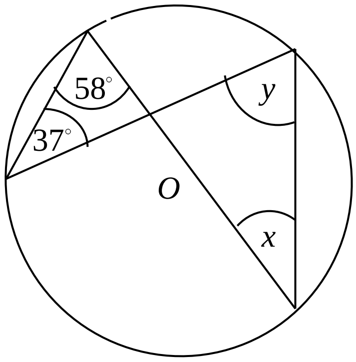
Theorem 3

Angles at the circumference of a circle standing on the same arc (or chord) are equal.

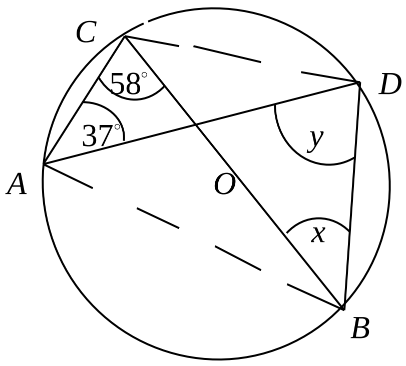


Example

Calculate the size of angles x and y, giving reasons for your answers.



Solution

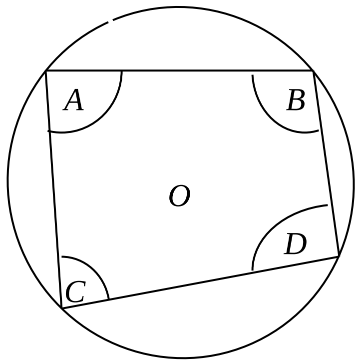


$x = 37^\circ$ and $y = 58^\circ$

Angles at the circumference of a circle standing on the same arc (or chord) are equal.

Theorem 4

The sum of the opposite angles of a cyclic quadrilateral is 180°.

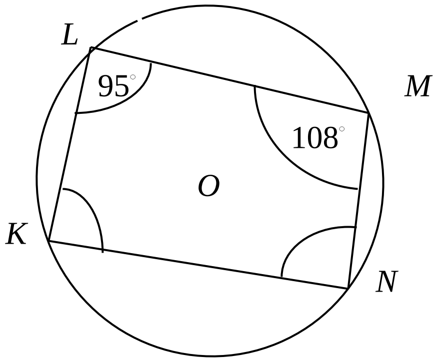


Here, $\hat{A} + \hat{D} = 180^\circ$ and $\hat{B} + \hat{C} = 180^\circ$

Note: $\hat{A} + \hat{B} + \hat{C} + \hat{D} = 360^\circ$

Example

In the cyclic quadrilateral $KLMN$, $\hat{L} = 95^\circ$ and $\hat{M} = 108^\circ$. Evaluate K and N, giving reasons for your answers.



Here, $K + M = 180^\circ$

So, $K + 108^\circ = 180^\circ$

So, $K = 180^\circ - 108^\circ$

$\therefore$ $K = 72^\circ$

Also, $L + N = 180^\circ$

So, $95^\circ + N = 180^\circ$

So, $95^\circ + N = 180^\circ$

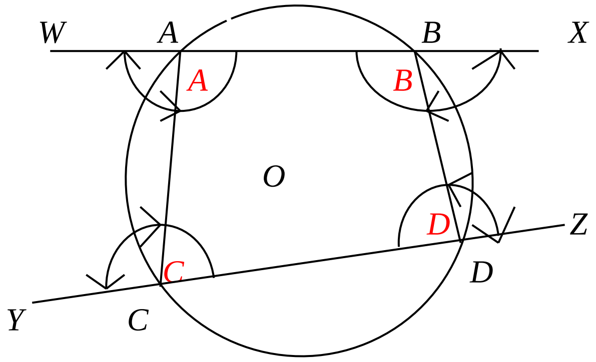
So, $N = 180^\circ - 95^\circ$

$\therefore \quad N = 85^\circ$

The sum of the opposite angles of a cyclic quadrilateral is 180°.

Theorem 5

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



Here, $W\hat{A}C = D$

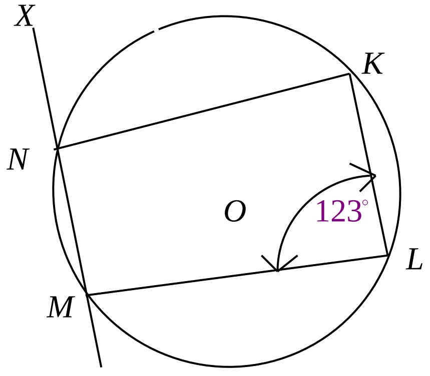
Also, $X\hat{B}D = C$

Also, $Y\hat{C}A = B$

Also, $Z\hat{D}B = A$

Example

In the cyclic quadrilateral $KLMN$, angle $KLM = 123^\circ$. Evaluate angles XNK and KNM, stating reasons for your answers.



Solution

$XNK = 123^{\circ}$ since the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

$KNM + L = 180^{\circ}$ since the sum of the opposite angles of a cyclic quadrilateral is 180°.

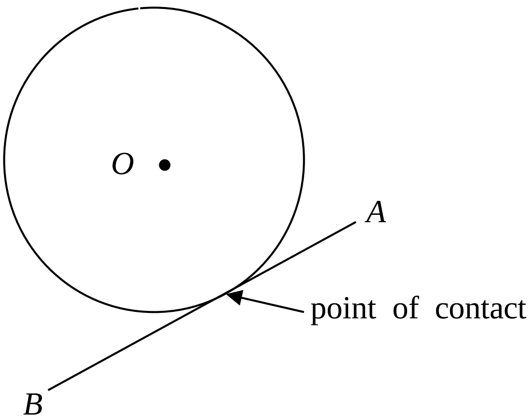
So, $KNM + 123^{\circ} = 180^{\circ}$

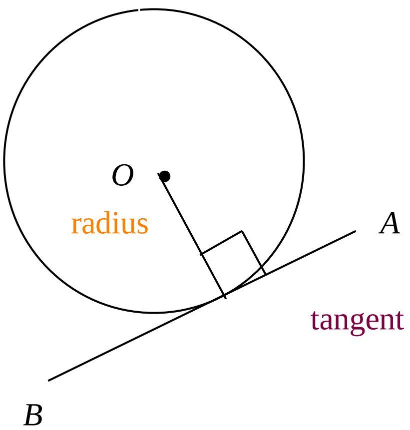
Now, $KNM = 180^{\circ} - 123^{\circ}$

$\therefore \quad KNM = 57^{\circ}$

Tangent to a circle (Theorem 6)

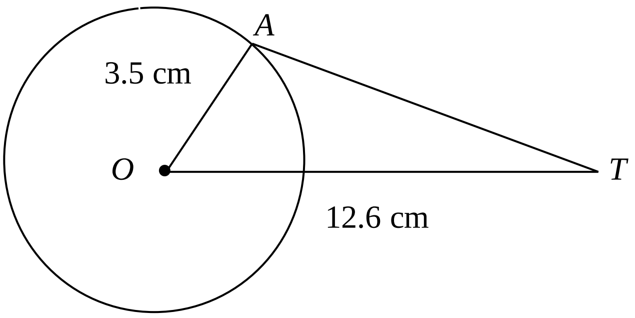
A tangent to a circle is a straight line drawn outside the circle in such a way that the line touches the circle ONLY at one point on the circumference of the circle. This point is called the point of contact or point of tangency.





The angle formed by a radius and a tangent at a point of contact is 90°.

Example



In the diagram above, TA is a tangent to the circle and OT is a straight line. If $OA = 3.5$ cm and $OT = 12.6$ cm, calculate

(a) the length of the tangent TA (b) $O\hat{T}A$

Solution

(a) Since TA is a tangent to the circle, then ΔOAT is right angled

Now, $OT^2 = OA^2 + TA^2$

So, $12.6^2 = 3.5^2 + TA^2$

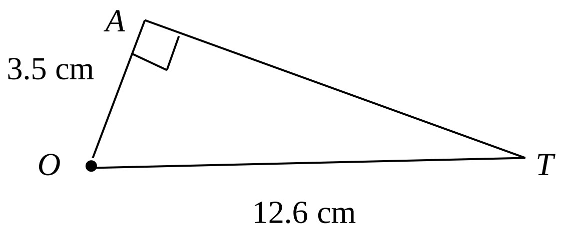
Then $TA^2 = 12.6^2 - 3.5^2 = 158.76 - 12.25$

So, $TA^2 = 146.51$

Now, $TA = \sqrt{146.51}$

$\therefore$ $TA = 12.1$ cm

(b)



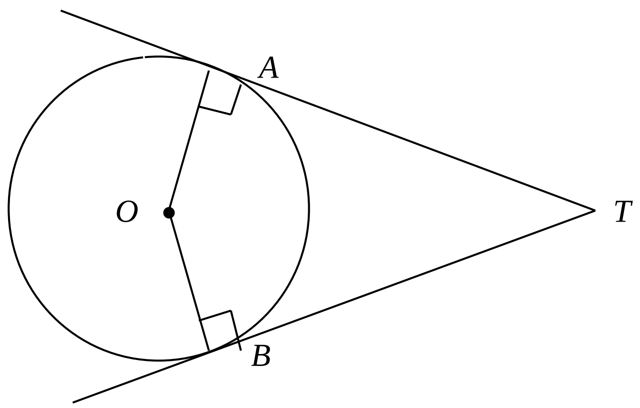
Let $O\hat{T}A = \theta$

Now, $\sin\theta = \dfrac{3.5}{12.6}$

So, $\theta = \sin^{-1}\left(\dfrac{3.5}{12.6}\right) = \sin^{-1} 0.278$

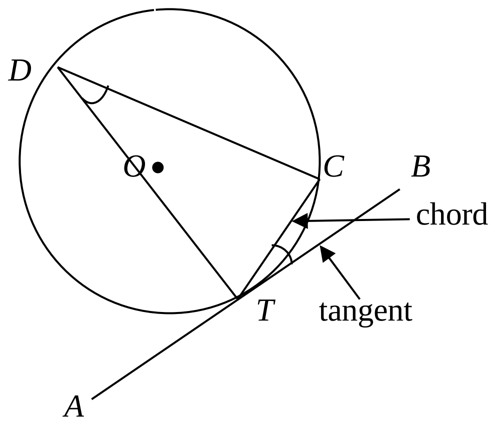
$\therefore$ $O\hat{T}A = 16.1^\circ$

Note: The tangents to a circle from an external point are equal in length.



Here, $TA = TB$

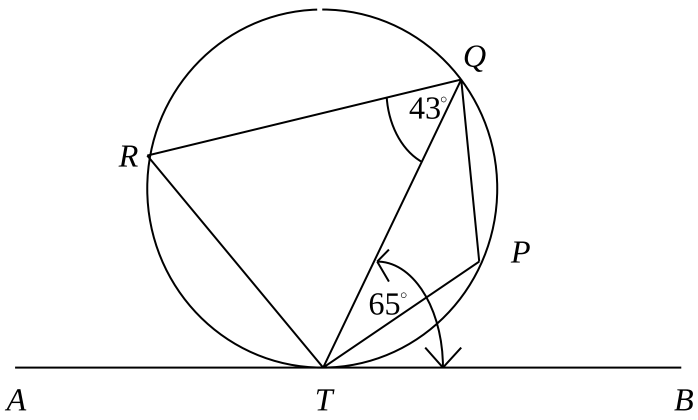
The angle between a tangent and a chord (Theorem 7)



The angle between a tangent to a circle and a chord at the point of contact is equal to the angle in the alternate segment of the circle.

Example

In the diagram below, ATB is a tangent to a circle and $PQRT$ is a cyclic quadrilateral.



Given that $R\hat{Q}T = 43^\circ$ and $Q\hat{T}B = 65^\circ$, evaluate

(a) $Q\hat{R}T$ (b) $Q\hat{P}T$ (c) $R\hat{T}A$

Solution

(a) $Q\hat{R}T = Q\hat{T}B$

$Q\hat{R}T = 65^\circ$

(b) Since $PQRT$ is a cyclic quadrilateral, $Q\hat{R}T + Q\hat{P}T = 180^\circ$

So, $65^\circ + Q\hat{P}T = 180^\circ$

Now, $Q\hat{P}T = 180^\circ - 65^\circ$

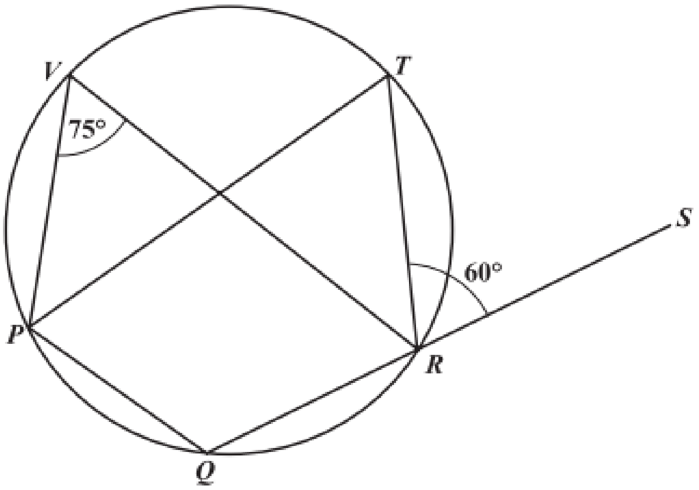
$\therefore$ $Q\hat{P}T = 115^\circ$

(c) $R\hat{T}A = R\hat{Q}T$

$R\hat{T}A = 43^\circ$

Example (January 2019)

The diagram below, **not drawn to scale**, shows a circle. The points *P*, *Q*, *R*, *T* and *V* are on the circumference. *QRS* is a straight line. Angle *PVR* = 75° and angle *TRS* = 60°.



Determine the value of EACH of the following angles. Show detailed working where necessary and give a reason to support your answer.

(i) Angle *PTR* **(2 marks)**

(ii) Angle *TPQ* **(2 marks)**

(iii) Obtuse angle *POR* where *O* is the centre of the circle. **(2 marks)**

Solution

(i) Angle $PTR = \angle PVR = 75^\circ$

Angles subtended on the circumference of a circle by the same arc (or chord) are equal.

(ii) Angle $TPQ = 60^\circ$

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

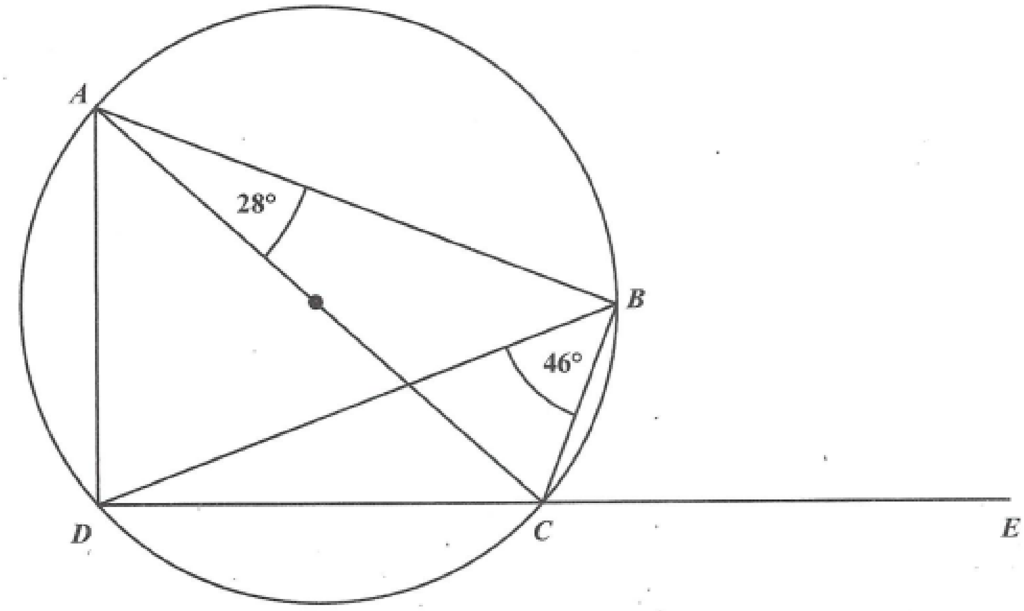
(iii) Angle $POR = 2 \times 75^\circ$

$\therefore$ Angle $POR = 150^\circ$

The angle at the centre of a circle is twice the angle at the circumference standing on the same arc (or chord).

Example (May 2019)

The diagram below shows a circle where AC is a diameter. B and D are two other points on the circle and DCE is a straight line. Angle $CAB = 28°$ and $\angle DBC = 46°$.



Calculate the value of each of the following angles. Show detailed working where necessary and give a reason to support your answers.

(i) $\angle DBA$ **(2 marks)**

(ii) $\angle DAC$ **(2 marks)**

(iii) $\angle BCE$ **(2 marks)**

Solution

(i) Chords AB and BC meet within a semi-circle.

So, $\angle ABC = 90^\circ$

Now, $\angle DBA = 90^\circ - 46^\circ$

$\therefore \angle DBA = 44^\circ$

The angle where two chords meet within a semi-circle is 90°

(ii) $\angle DAC = \angle DBC = 46^\circ$

Angles subtended on the circumference of a circle by the same arc (or chord) are equal.

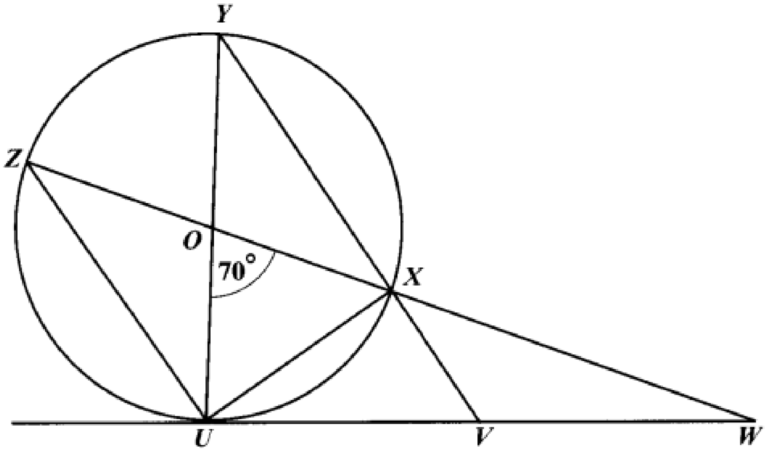
(iii) $\angle BCE = \angle DAC + \angle CAB = 46^\circ + 28^\circ$

$\angle BCE = 74^\circ$

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Example (May 2012)

The diagram below, **not drawn to scale**, shows a circle, centre *O*. The line *UVW* is a tangent to the circle, *ZOXW* is a straight line and angle *UOX* = 70°.



Calculate, showing working where necessary, the measure of angle

a) *OUZ* **(2 marks)**

b) *UVY* **(3 marks)**

c) *UWO*. **(2 marks)**

Solution

(a) $\angle ZOU = 180^\circ - 70^\circ$, since $ZOXW$ is a straight line.

So, $\angle ZOU = 110^\circ$

For triangle ZOU, $ZO = OU$, since ZO and OU are two radii of the circle. Now, triangle ZOU is an isosceles triangle.

Now, $\angle OUZ = \dfrac{180^\circ - 110^\circ}{2} = \dfrac{70^\circ}{2}$

$\therefore$ $\angle OUZ = 35^\circ$

(b) The radius OU meets the tangent UVW at U. So, $\angle YUV = 90^\circ$

Now, triangle YUV is a right-angled triangle.

Since the arc UX subtends an angle of 70° at the centre of the circle and angle UYX on the circumference of the circle, then $\angle UYX = \dfrac{70^\circ}{2}$

So, $\angle UYX = 35^\circ$

By considering the right-angled triangle YUV, $\angle UVY = 180^\circ - \left(90^\circ + 35^\circ\right)$

So, $\angle UVY = 180^\circ - 125^\circ$

$\therefore$ $\angle UVY = 55^\circ$

(c) $\angle UOW = 70^\circ$ and $\angle OUW = 90^\circ$

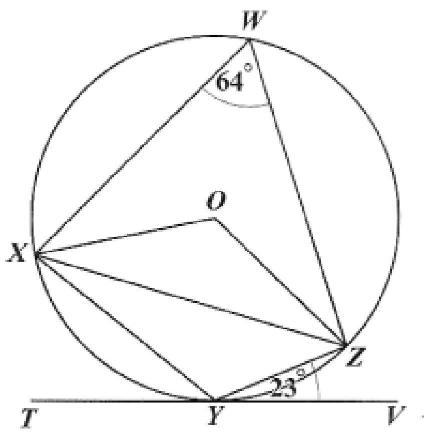
By considering the right-angled triangle OUW, $\angle UWO = 180^\circ - \left(90^\circ + 70^\circ\right)$

Now, $\angle UWO = 180^\circ - 160^\circ$

$\therefore$ $\angle UWO = 20^\circ$

Example (May 2011)

In the diagram below, **not drawn to scale**, W, X, Y and Z are points on the circumference of a circle, centre O. TYV is a tangent to the circle at Y, $\angle XWZ = 64°$ and $\angle ZYV = 23°$.



Calculate, **giving reasons for your answer**, the measure of angle

(i) XYZ **(2 marks)**

(ii) YXZ **(2 marks)**

(iii) OXZ. **(3 marks)**

Solution

(i) $WXYZ$ is a cyclic quadrilateral.

So, $\angle XWZ + \angle XYZ = 180^\circ$

So, $\angle XYZ = 180^\circ - 64^\circ$

$\therefore \angle XYZ = 116^\circ$

The sum of the opposite angles of a cyclic quadrilateral is 180°

(ii) $\angle YXZ = 23^\circ$

The angle between a tangent to a circle and a chord at the point of contact is equal to the angle in the alternate segment of the circle.

(iii) OX and OZ are two radii of the circle.

So, triangle OXZ is an isosceles triangle, since $OX = OZ$

Now, chord XZ subtends $\angle XWZ$ on the circumference and $\angle XOZ$ at the centre of the circle.

So, $\angle XOZ = 2 \times 64^\circ$

$\therefore \angle XOZ = 128^\circ$

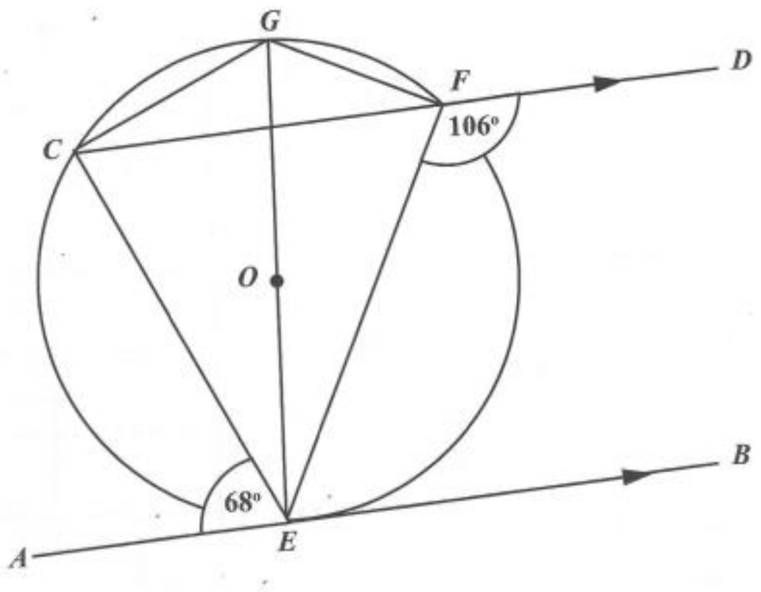
Since triangle OXZ is an isosceles triangle, then $\angle OXZ = \dfrac{180^\circ - 128^\circ}{2} = \dfrac{52^\circ}{2}$

$\therefore \angle OXZ = 26^\circ$

The angle formed at the centre of a circle by an arc (or chord) is twice the angle formed at the circumference by the same arc (or chord)

Example (May 2021)

In the diagram below, E, C, G and F are points on the circumference of a circle. EG is a diameter of the circle. The tangent AEB is parallel to CD. Angle $AEC = 68°$ and angle $EFD = 106°$.



Determine the value of EACH of the following angles. Show detailed working where necessary and give a reason to support your answer.

(i) ECD **(2 marks)**

(ii) CEG **(2 marks)**

(iii) CGF **(2 marks)**

Solution

(i) $ECD = 68^\circ$

The lines AEB and CD are parallel. So, angles AEC and ECD are alternate angles.

(ii) Since the radius OE meets the tangent AEB at the point E, then angle $OEA = 90^\circ$.

Now, angle $CEG = 90^\circ - 68^\circ$

So, angle $CEG = 22^\circ$

(iii) Since the lines AEB and CD are parallel, the angles EFD and AEF are alternate angles. So, angle $AEF = 106^\circ$.

Now, angle $CEF = 106^\circ - 68^\circ = 38^\circ$

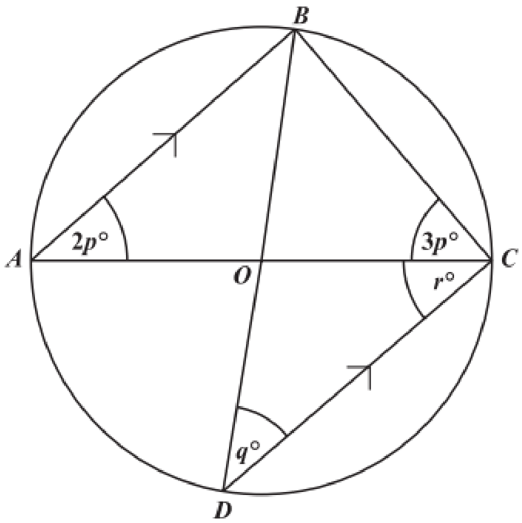
Now, $ECGF$ is a cyclic quadrilateral and the sum of opposite angles of a cyclic quadrilateral is 180°.

Now, angle $CGF = 180^\circ - 38^\circ$

So, angle $CGF = 142^\circ$

Example (January 2021)

In the diagram below, A, B, C and D are points on the circumference of a circle, with centre O. AOC and BOD are diameters of the circle. AB and DC are parallel.



(i) State the reason why angle ABC is 90°. **(1 mark)**

(ii) Determine the value of EACH of the following angles. Show detailed working where necessary and give a reason to support your answer.

a) Angle BAC **(2 marks)**

b) Angle q **(2 marks)**

(iii) Calculate the value of angle r. **(1 mark)**

<u>Solution</u>

(i) Angle ABC is 90° because angle ABC is where the chords AB and BC meet within a semi-circle.

(ii) (a) Since angle ABC is 90°, then triangle ABC is right-angled.

Now, angle BAC + angle $BCA = 90^\circ$

That is, $2p + 3p = 90^\circ$

So, $5p = 90^\circ$

Now, $p = \dfrac{90^\circ}{5} = 18^\circ$

Now, angle $BAC = 2 \times 18^\circ$

So, angle $BAC = 36^\circ$

(b) $q = 36^\circ$

The arc BC subtends an angle of 36° at the point A on the circumference. The arc BC also subtends the angle q at the point D on the circumference. Angles subtended at the circumference of a circle by the same arc (or chord) are equal.

(iii) Since the chords BC and CD meet within a semi-circle, then angle $BCD = 90^\circ$

Also, $3p = 3 \times 18^\circ = 54^\circ$

Now, $r = 90^\circ - 54^\circ$

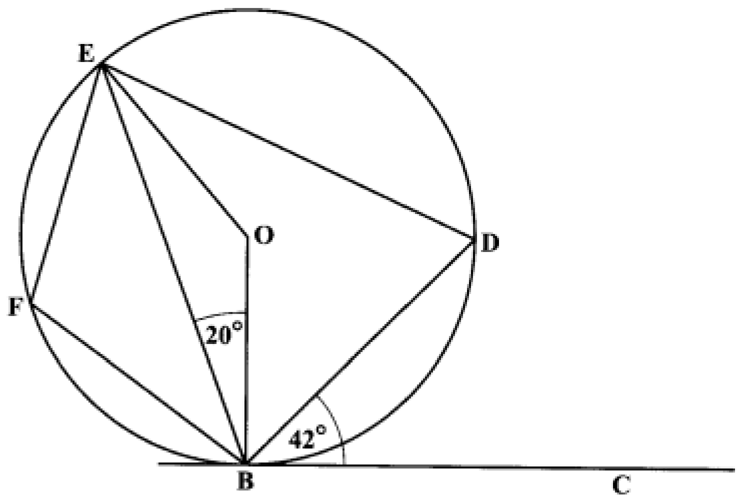
So, $r = 36^\circ$

OR Since OC and OD are radii of the circle, then triangle OCD is an isosceles triangle. So, angle $r =$ angle q

So, $r = 36^\circ$

TRY THIS #7 (May 2014)

The diagram below, **not drawn to scale**, shows a circle, centre O. The line BC is a tangent to the circle at B. Angle $CBD = 42°$ and angle $OBE = 20°$.



Calculate, giving a reason for EACH step of your answer, the measure of:

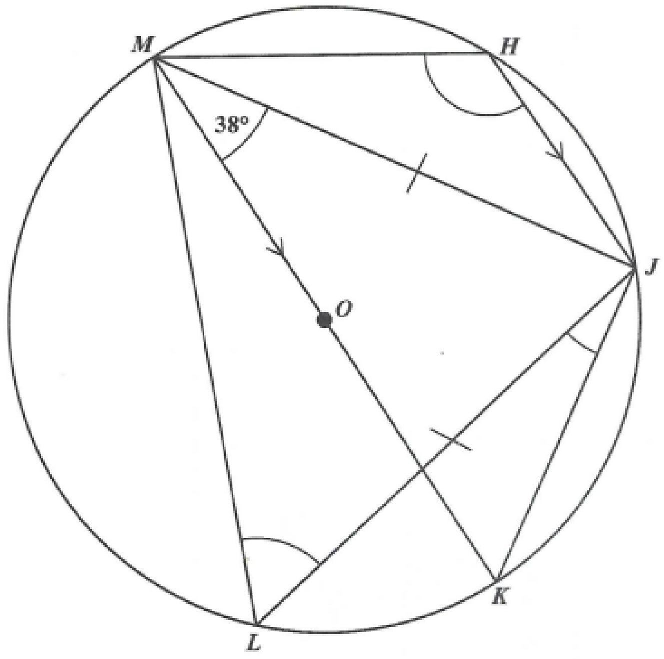
(i) $<BOE$ **(2 marks)**

(ii) $<OED$ **(2 marks)**

(iii) $<BFE$ **(3 marks)**

TRY THIS #8 (May 2022)

H, J, K, L and M are points on the circumference of a circle with centre O. MK is a diameter of the circle and it is parallel to HJ. $MJ = JL$ and angle $JMK = 38°$.



(i) Explain, giving a reason, why angle

a) $HJM = 38°$ **(1 mark)**

b) $MJK = 90°$. **(1 mark)**

(ii) Determine the value of EACH of the following angles. Show detailed working where appropriate.

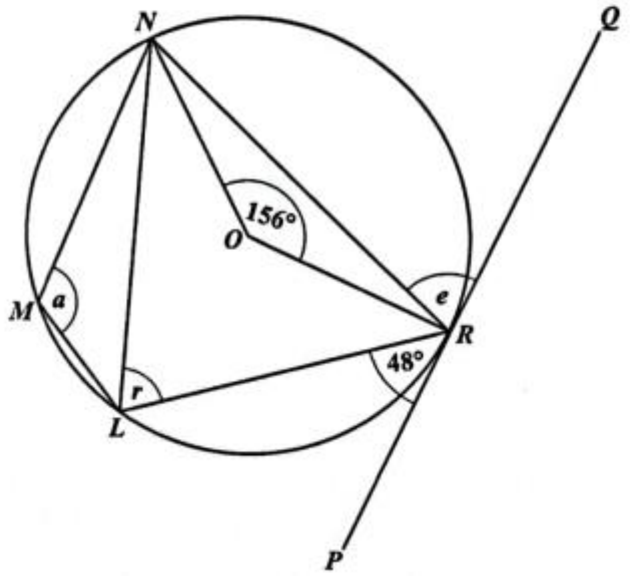
a) Angle MLJ **(2 marks)**

b) Angle LJK **(1 mark)**

c) Angle JHM **(1 mark)**

TRY THIS #9 (May 2023)

L, *M*, *N* and *R* are points on the circumference of a circle, with centre *O*. *PQ* is a tangent to the circle at *R*. Angle $PRL = 48°$ and Angle $RON = 156°$.



Find the value of EACH of the following angles, giving reasons for EACH of your answers. Show ALL working where appropriate.

(i) Angle *r* **(2 marks)**

(ii) Angle *e* **(2 marks)**

(iii) Angle *a* **(2 marks)**

END OF MARATHON SESSION

ALL THE BEST ON MONDAY MAY 13

MARATHON BOOK CREATED BY **MR. RICARDO BARKER**

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